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Formation-containment control for general linear multi-agent systems with time-varying delays and switching topologies

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Abstract— **Formation-containment control problems for general linear multi-agent systems with time-varying delays and switching topologies are studied. The leaders are required to accomplish a given time-varying formation and the followers are allowed to enter the convex envelope spanned by those of the leaders simultaneously. Firstly, formation-containment protocols based on distributed state observer with switching interaction topologies and time-varying delays are presented for leaders and followers respectively, where an edge-based state observer is presented for each follower to estimate the whole states of all the leaders under the influences of switching interaction topologies and time-varying delays. Then, formation-containment problems are transformed into asymptotic stability problems. Furthermore, an algorithm to determine the gain matrices in the protocols is given based on linear matrix inequality technique and common Lyapunov–Krasovskii stability theory. Sufficient conditions for multi-agent systems to achieve formation-containment under the designed protocol are proposed. Finally, numerical simulations are provided to demonstrate theoretical results.**

Index Terms— **Formation-containment control; general linear multi-agent systems; time-varying delays; switching topologies**

I. Introduction

In recent years, cooperative control of multi-agent systems (MASs) has been widely applied in different fields. Its research field could be divided into different categories, where consensus control [1], [2], formation control [3], [4], containment control [5], [6], and formation-containment control [7] are four branches of the most attracted ones.

Formation control problem is a fundamental problem and has been studied a lot during the past decades. With the extensive study in consensus-based theory[8], [9], more and more researches try to use consensus-based methods to solve the formation problems. However, in order to meet the practical environment such as the congestion of the interaction channel and communication range constraints, communication delays and switching topologies emerge. By using the state information of each agent and neighboring agents, Dong et al. in [10] constructed a protocol for general-linear MASs with time-varying delays. Using local information of the neighboring agents, Xiao et al. in [11] proposed an ideal protocol to solve the time-varying formation control problem for MASs with both time-varying delays and switching interaction topologies, which means formation control has been studied in a more practical field.

In containment control problem, agents are classified into leaders and followers. The aim of containment control is that followers are required to converge to the convex hull spanned by the leaders. Ji in [12] proposed a hybrid "stop-go" control strategy for first-order MASs to achieve containment. Containment problems with time-varying delays or stochastic topologies were discussed in [13], [14]. However, in containment control, there is no interaction among the leaders. In some practical scenarios, the leaders are allowed to achieve a desired formation and and the followers should enter the convex envelope formed by the leaders. Based on formation control and containment control, a more interesting research named formation-containment problem arises. Time-varying delays on formation-containment control were considered in [15]. Therefore, formation-containment problems under the influence of both time-varying delays and switching interaction topologies are still open.

Motivated by the above facts, formation-containment control problems for general linear MASs with time-varying delays and switching interaction topologies are investigated. Compared with the previous researches on formation-containment, the new contributions of this paper are twofold. First, formation-containment control can be accomplished with both time-varying delays and switching topologies for general-linear MASs. In [15], only time-varying delays are considered. Second, due to the dynamic characteristics of the leaders and the impact of the formation on the followers, the formation-containment control problem cannot be directly decoupled into the formation control and containment control problems. In contrast to formation control problems in [10], [11], the states of the followers are able to stay in the convex envelope formed by leaders. While in [7], [14], they only studied the containment control problems. The difficulty of formation-containment control is greater than the sum of the above mentioned formation and containment control.

Throughout this paper, I represents an identify matrix with appropriate size. $\otimes$ denotes Kronecker product. $\boldsymbol{1}$ is used to describe a column vector with 1 as its element. $diag\{D_1, \cdots, D_N\}$ is used to present a diagonal matrix.

II. Preliminaries and problem description

In this section, the basic graph theory is given and the statement of problem is proposed.

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A. Preliminaries

A weighted undirected graph G can be represented by $\{V,T,W\}$, where $V = \{v_1, v_2, \cdots v_N\}$ is a set of nodes, $T \subseteq \{(v_i, v_j) : v_i, v_j \in V\}$ is the set of edges, and $W = [a_{ij}] \in \mathbb{R}^{N\times N}$ is a weighted adjacency matrix. Let $e_{ij} = (v_i, v_j)$ denotes the edge of G and w_{ij} denotes the nonnegative element with e_{ji}. Define $w_{ij} > 0$ if and only if $e_{ji} \in T$ and $w_{ij} = 0$ otherwise. $N_i = \{v_j \in V : (v_j, v_i) \in T\}$ is the set of neighbors of node v_i. The Laplacian matrix L is defined as $L = D - W$, where $D = diag\left\{\sum_{j=1}^{N} w_{1j}, \sum_{j=1}^{N} w_{2j}, \cdots, \sum_{j=1}^{N} w_{Nj}\right\}$. A path from node v_{i1} to v_{ik} is a series of ordered edges (v_{i1}, v_{i2}), (v_{i2}, v_{i3}), $\cdots$, (v_{ik-1}, v_{ik}). The definition of undirected graph is that $v_{ij} \in T$ implies $v_{ji} \in T$ and $w_{ij} = w_{ji}$. The undirected graph is said to be connected if there is a path between any distinct pair of nodes.

Definition 1: An agent is called a leader if it has no neighbors and a follower if it has at least a neighbor.

Assumption 1: Each switching topology $G^F_{\sigma(t)}$ among the followers is connected and undirected.

Assumption 2: There exists at least one follower which has access to all leaders' states at each possible graph.

Lemma 1: ([17]) If G is connected, then L has a simple 0 eigenvalue with $1_N \big/ \sqrt{N}$ as its right eigenvector, and all the other eigenvalues are positive.

The index set of all the switching graphs is represented by $\mathrm{H} = \{1, 2, \cdots, h\}$. There exists an infinite sequence of non-overlapping time intervals $[t_k, t_{k+1})$ with $t_0 = 0$, $t_k - t_{k+1} \geqslant T_d > 0$. T_d is said as the dwell time, during which the graph keeps fixed. The graph changes at switching sequence t_{k+1}. Let $\sigma(t) : [0, \infty) \to \{1, 2, \cdots, h\}$ denotes a switching signal. $G_{\sigma(t)}$ and $L_{\sigma(t)}$ represent the graph and Laplacian matrix at t. Let $L^F_{\sigma(t)}$ and $L^F_{\sigma(t)}$ denote Laplacian matrix among the followers and leaders.

B. Problem description

Consider a MAS with M leaders and N followers. Let $O_E = \{1, 2, \cdots, M\}$ and $O_F = \{M+1, M+2, \cdots, M+N\}$ denote the leader set and follower set, respectively. The dynamics of the ith agent can be expressed as

$$\begin{cases} \dot{z}_i(t) = Az_i(t) + Bu_i(t) & i \in O_E \\ \dot{x}_i(t) = Ax_i(t) + Bu_i(t) & i \in O_F \end{cases} \tag{1}$$

where $z_i(t)$ and $u_i(t)\,(i \in O_E)$ are the state and control input of the ith leader, $x_i(t)$ and $u_i(t)\,(i \in O_F)$ are the state and control input of the ith follower.

Definition 2: A time-varying formation is specified by $h_E(t) = [h_1^T(t), h_2^T(t), \cdots, h_M^T(t)] \in \mathbb{R}^{Mn}$. Leaders in MAS(1) are said to realize time-varying formation $h_E(t)$ if there exists a vector-valued function $r(t) \in \mathbb{R}^n$ such that

$$\lim_{t\to\infty} \left(z_i(t) - r(t) - h_i(t)\right) = 0 \;\; (i = 1, 2, \cdots, M) \tag{2}$$

Where $r(t)$ is called a formation reference function.

Definition 3: MAS described by (1) is said to achieve state containment if there exist nonnegative constants $\rho_{ij}\,(i \in O_F, j \in O_E)$ satisfying $\sum_{j=1}^{M} \rho_{ij} = 1$, underlying any bounded initial states, such that

$$\lim_{t\to\infty} \left(x_i(t) - \sum\nolimits_{j=1}^{M} \rho_{ij} z_j(t)\right) = 0 \tag{3}$$

Definition 4: MAS (1) is said to realize formation-containment if for any give bounded initial states, there is a vector-valued function $r(t) \in \mathbb{R}^n$ and nonnegative $\rho_{ij}\,(i \in O_F, j \in O_E)$ satisfying $\sum_{j=1}^{M} \rho_{ij} = 1$ so that for any $i \in O_F$ and $j \in O_E$ (2) and (3) hold simultaneously.

Consider the following formation-containment protocol with time-varying delays and switching topologies.

for $i \in O_E$

$$\begin{aligned} u_i(t) = {} & K_1 z_i(t) + K_2 \sum_{j\in N^i_{\sigma(t)}} w_{ij} \left[(z_j(t-\tau(t)) \right. \\ & \left. -h_j(t-\tau(t))) - (z_i(t-\tau(t)) - h_i(t-\tau(t)))\right] \end{aligned} \tag{4}$$

for $i \in O_F$

$$\begin{cases} u_i(t) = K_3 x_i(t) - K_4 \sum\limits_{j=1}^{M} \rho_{i,j} \hat{\xi}_{i,j}(t) \\ \dot{\hat{\xi}}_i(t) = \left(I_M \otimes (A + BK_1)\right) \hat{\xi}_i(t) \\ \quad - K_5 \left[b_i^{\sigma(t)} \left(\hat{\xi}_i(t-\tau(t)) - z(t-\tau(t))\right) \right. \\ \quad \left. + \sum\limits_{k=M+1}^{M+N} w_{ik} (\hat{\xi}_i(t-\tau(t)) - \hat{\xi}_k(t-\tau(t))) \right] \end{cases} \tag{5}$$

where $\hat{\xi}_i(t) = [\hat{\xi}^T_{i,1}(t), \hat{\xi}^T_{i,2}(t), \cdots, \hat{\xi}^T_{i,M}(t)]^T$ with $\hat{\xi}_{i,j}(t)$ representing the ith follower's estimate for the state of jth leader. $z(t) = [z_1^T(t), z_2^T(t), \cdots, z_M^T(t)]^T$. $K_i(i = 1, 2, 3, 4, 5)$ are the constant gain matrices. ρ_{ij} is the predefined nonnegative constant satisfying $\sum_{j=1}^{M} \rho_{ij} = 1$, and the nonnegative $b_i^{\sigma(t)} > 0$ if and only if follower i has access to all leaders' states at t, otherwise $b_i^{\sigma(t)} = 0$. $\tau(t)$ is the time-varying delay.

Assumption 3: $0 \leqslant \tau(t) \leqslant \sigma$ and $|\dot{\tau}(t)| \leqslant \delta < 1$, where σ and δ are known constants.

The following lemmas are useful for analyzing the formation-containment problem.

Lemma 2: ([18]) Let $\eta(t) \in \mathbb{R}^{2d}$ be a vector-valued function with first-order continuous-derivative entries. The following integral inequality holds:

$$\begin{aligned} & -\int_{t-\tau(t)}^{t} \dot{\eta}^T(s) S \dot{\eta}(s) ds \\ & \quad \leq \varsigma^T(t) \begin{bmatrix} M_1^T + M_1 & -M_1^T + M_2 \\ * & -M_2^T - M_1 \end{bmatrix} \varsigma(t) \\ & \quad + \tau(t)\varsigma^T(t) \begin{bmatrix} M_1^T \\ M_2^T \end{bmatrix} S^{-1} [M_1, M_2] \varsigma(t) \end{aligned} \tag{6}$$

where $M_1, M_2 \in \mathbb{R}^{2d}$, $S = S^T > 0$, $\varsigma(t) = [\eta^T(t), \eta^T(t-\tau(t))]^T$, superscript $*$ is an item derived from symmetry $-M_1 + M_2^T$.

Let $H_{\sigma(t)} = B_{\sigma(t)} + L_F^{\sigma(t)}$, $\sigma(t) \in \{1, 2, \cdots, p\}$, where $B_{\sigma(t)}$ is a diagonal matrix which diagonal element is 0 or 1.

Lemma 3: ([11]) If each switching topology $G^F_{\sigma(t)}$ is connected, then the symmetric matric $H_{\sigma(t)}$ at t associated with $G^F_{\sigma(t)}$ is positive define.

Let, $\bar{\lambda}_1 = \min\left\{\lambda^i_{\sigma(t)}\right\}$, $\bar{\lambda}_2 = \max\left\{\lambda^i_{\sigma(t)}\right\}$, $\sigma(t) \in \{1, 2, \cdots, p\}$, $\lambda^i_{\sigma(t)}$ is the eigenvalue of real symmetric positive definite matrix.

Lemma 4: ([11]) For any $i, \sigma(t) \in \{1, 2, \cdots, p\}$, $\Theta^i_{\sigma(t)} = \Phi_0 + \lambda^i_{\sigma(t)}\Phi_1 < 0$ if and only if $\Theta_i = \Phi_0 + \bar{\lambda}_i\Phi_1 < 0$ ($i \in \{1, 2\}$).

In the current paper, the following two problems for MAS (1) with time-varying delays and switching topologies are investigated. First, how to design the protocols and state observer. Second, under what condition the formation-containment can be achieved.

III. Main results

In this section, a novel formation-containment control framework is put forward to handle MASs with time-varying delays and switching interaction topologies. Firstly, the formation-containment are transformed into asympotic stability problems. Secondly, an Algorithm is proposed to design the protocols and state observer. Finally, the stability of the system and the convergence of observing are proved.

The Laplacian matrix $L_{\sigma(t)}$ is given as

$$L_{\sigma(t)} = \begin{bmatrix} L^E_{\sigma(t)} & 0 \\ L^{FE}_{\sigma(t)} & L^F_{\sigma(t)} \end{bmatrix}$$

where $L^E_{\sigma(t)} \in \mathbb{R}^{M\times M}$, $L^{FE}_{\sigma(t)} \in \mathbb{R}^{N\times M}$, and $L^F_{\sigma(t)} \in \mathbb{R}^{N\times N}$.

Under protocol (4) the dynamic of closed-loop of leaders' can be described

$$\begin{aligned}&\dot{z}(t) = (I_M \otimes (A + BK_1))\, z(t)\\ &-\left(L^E_{\sigma(t)} \otimes BK_2\right)(z(t-\tau(t)) - h(t-\tau(t)))\end{aligned} \quad (7)$$

Under protocol (5) the closed-loop of followers' can be written as

$$\dot{x}_i(t) = (A + BK_3)x_i(t) - BK_4\sum_{j=1}^{M}\rho_{ij}\hat{\xi}_{i,j}(t) \;\; i \in O_F \quad (8)$$

Let $\tilde{\xi}_{i,j}(t) = \hat{\xi}_{i,j}(t) - z_j(t)$ denotes the observing error between ith observer and jth leader. Then (8) could be rewritten as follows

$$\begin{aligned}\dot{x}_i(t) =& (A + BK_3)x_i(t) - BK_4\sum_{j=1}^{M}\rho_{ij}\tilde{\xi}_{i,j}(t)\\ &- BK_4\sum_{j=1}^{M}\rho_{ij}z_j(t)\end{aligned} \quad (9)$$

Let $\tilde{z}_i(t) = z_i(t) - h_i(t)\,(i \in O_E)$, $\tilde{z}(t) = \left[\tilde{z}_1^T(t), \tilde{z}_2^T(t), \cdots \tilde{z}_M^T(t)\right]^T$. It follows from (9) that

$$\begin{aligned}\dot{\tilde{z}}(t) =& (I_M \otimes (A + BK_1))\,\tilde{z}(t)\\ &-\left(L^E_{\sigma(t)} \otimes BK_2\right)\tilde{z}(t-\tau(t))\\ &+(I_M \otimes (A + BK_1))\,h(t) - (I_M \otimes I_n)\,\dot{h}(t)\end{aligned} \quad (10)$$

Let $U = [\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_M]$ be an orthogonal constant matrix, where $\bar{u}_1 = 1_M/\sqrt{M}$, then form *Lemma 1*, one could get $U^T L^E_{\sigma(t)} U = diag(0, \tilde{U}^T L^E_{\sigma(t)}\tilde{U})$, where $\tilde{U} = [\bar{u}_2, \cdots, \bar{u}_M]$. Let $\theta(t) = (\bar{u}_1^T \otimes I_n)\tilde{z}(t)$ and $\phi(t) = (\tilde{U}^T \otimes I_n)\tilde{z}(t)$, then the leaders' system can be divided into two parts.

$$\begin{aligned}\dot{\theta}(t) =& \tilde{A}\theta(t) - \tfrac{1}{\sqrt{N}}\left(1_N^T \otimes I_n\right)\dot{h}(t)\\ &+\tfrac{1}{\sqrt{N}}\left(1_N^T \otimes (A + BK_1)\right)h(t)\end{aligned} \quad (11)$$

$$\begin{aligned}\dot{\phi}(t) =& (I_{M-1} \otimes \tilde{A})\phi(t)\\ &-(\tilde{U}^T L^E_{\sigma(t)}\tilde{U} \otimes BK_2)\phi(t-\tau(t))\\ &+(\tilde{U}^T \otimes (A + BK_1))h(t) - (\tilde{U}^T \otimes I_n)\dot{h}(t)\end{aligned} \quad (12)$$

Where $\tilde{A}$=A+BK_1.

Lemma 5: [16] The leaders of MAS (1) achieve the time-varying formation specified by $h_E(t)$. If and only if

$$\lim_{t\to\infty}\phi(t) = 0 \quad (13)$$

An Algorithm with five steps is proposed to design the protocols and observer.

Algorithm 1. For MAS(1) to achieve the formation-containment, $K_i (i = 1, 2, 3, 4, 5)$ can be designed in these steps.

Step 1: Choose suitable K_1 to assign the eigenvalue of $A + BK_1$ at the closed left-half complex plane. If (A, B) is controllable, we always have an appropriate K_1.

Step 2: Solve LMI (14). If there exist positive symmetric matrices R_E, Ω_E, S_E and real matrix $\bar{K}_2$ for any $\bar{\lambda}^E_i (i = 1, 2)$, LMI (14) is feasible, the gain matric $K_2 = \bar{K}_2\Omega^{-1}$; otherwise , the Algorithm stops.

$$\prod(\bar{\lambda}^E_i) = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & 0 & R_E \\ * & \Xi_{22} & \Xi_{23} & \sigma S_E & 0 \\ * & * & -\sigma S_E & 0 & 0 \\ * & * & * & -\sigma S_E & 0 \\ * & * & * & * & -\Omega_E \end{bmatrix} < 0 \quad (14)$$

where

$$\begin{aligned}&\Xi_{11} = R_E\tilde{A}^T + \tilde{A}R_E - \bar{\lambda}^E_i(B\bar{K}_2) - \bar{\lambda}^E_i(B\bar{K}_2)^T - (1-\delta)\Omega_E,\\ &\Xi_{12} = R_E - \bar{\lambda}^E_i B\bar{K}_2 - (2-\delta)\Omega_E,\\ &\Xi_{13} = \sigma R_E\tilde{A}^T - \sigma\bar{\lambda}^E_i(B\bar{K}_2)^T,\\ &\Xi_{22} = -(3-\delta)\Omega_E,\\ &\Xi_{23} = -\sigma\bar{\lambda}^E_i(B\bar{K}_2)^T,\end{aligned}$$

Step 3: Choose suitable K_3 to assign the eigenvalue of $A + BK_3$ at the left-half complex plan.

Step 4: Choose the suitable K_4, following the equation K_4=$K_3 - K_1$.

Step 5: Solve LMI (15). If there exist positive symmetric matrices R_F, Ω_F, S_F and real matrix $\bar{K}_5$ for any $\bar{\lambda}^H_i (i = 1, 2)$, LMI (15) is feasible, the gain matric $K_5 = \bar{K}_5\Omega_F^{-1}$; otherwise , the Algorithm stops.

$$\prod(\bar{\lambda}^F_i) = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & 0 & R_F \\ * & \Xi_{22} & \Xi_{23} & \sigma S_F & 0 \\ * & * & -\sigma S_F & 0 & 0 \\ * & * & * & -\sigma S_F & 0 \\ * & * & * & * & -\Omega_F \end{bmatrix} < 0 \quad (15)$$

where

$$\begin{aligned}&\Xi_{11} = R_F\tilde{A}_F^T + \tilde{A}_F R_F - \bar{\lambda}^H_i\bar{K}_5 - \bar{\lambda}^H_i\bar{K}_5^T - (1-\delta)\Omega_F,\\ &\Xi_{12} = R_F - \bar{\lambda}^H_i\bar{K}_5 - (2-\delta)\Omega_F,\\ &\Xi_{13} = \sigma R_F\tilde{A}_F^T - \sigma\bar{\lambda}^H_i\bar{K}_5^T,\\ &\Xi_{22} = -(3-\delta)\Omega_F,\\ &\Xi_{23} = -\sigma\bar{\lambda}^H_i\bar{K}_5^T,\\ &\tilde{A}_F = I_M \otimes A.\end{aligned}$$

Based on the Algorithm 1, the following Theorem can be obtained.

Theorem 1: Using the protocols (4) and (5) designed by Algorithm 1, MAS described by (1) with time-varying delays and switching topologies can achieve the formation-containment if the following time-varying formation feasibility condition holds

$$(A+BK_1)\left(h_i(t)-h_j(t)\right)-\left(\dot{h}_i(t)-\dot{h}_j(t)\right)\equiv 0 \quad (16)$$

Proof:

Step 1: The leaders can achieve the desire formation.

Based on Lemma 5, let $\dot{\varphi}(t)=(I_{M-1}\otimes\tilde{A})\varphi(t)-(\tilde{U}^TL_{\sigma(t)}\tilde{U}\otimes BK_2)\varphi(t-\tau(t))$ denotes the switching system, then $\dot{\phi}(t)$ can be written as follow:

$$\dot{\phi}(t)=\dot{\varphi}(t)+(\tilde{U}^T\otimes(A+BK_1))h(t)-(\tilde{U}^T\otimes I_n)\dot{h}(t) \quad (17)$$

Construct the following common Lyapunov–Krasovskii candidate function

$$V_E(t)=V_{E1}(t)+V_{E2}(t)+V_{E3}(t) \quad (18)$$

where

$$\begin{aligned}V_{E1}(t)&=\varphi^T(t)\left(I_{M-1}\otimes R_E^{-1}\right)\varphi(t),\\ V_{E2}(t)&=\int_{t-\tau(t)}^{t}\varphi^T(s)\left(I_{M-1}\otimes\Omega_E^{-1}\right)\varphi(s)\,ds,\\ V_{E3}(t)&=\int_{-\sigma}^{0}\int_{t+\mu}^{t}\dot{\varphi}^T(s)\left(I_{M-1}\otimes S_E^{-1}\right)\dot{\varphi}(s)dsd\mu.\end{aligned}$$

Let $\Lambda_{\sigma(t)}=diag(\lambda_{\sigma(t)}^1,\lambda_{\sigma(t)}^2,\cdots,\lambda_{\sigma(t)}^N)$, as we mentioned that $\tilde{U}^TL_{\sigma(t)}^E\tilde{U}$ is symmetric, therefore , it is possible to find an orthogonal matrix $\tilde{U}_{\sigma(t)}$ which satisfying that $\tilde{U}_{\sigma(t)}^T\tilde{U}^TL_{\sigma(t)}^E\tilde{U}\tilde{U}_{\sigma(t)}=\Lambda_{\sigma(t)}^E$. Let $\eta(t)=\left(\tilde{U}_{\sigma(t)}^T\otimes I_n\right)\varphi(t)=\left[\eta_2^T(t),\eta_3^T(t),\cdots,\eta_M^T(t)\right]^T$, and $\hat{\eta}_i(t)=\left[\eta_i^T(t),\eta_i^T(t-\tau(t))\right]^T$, then taking the time derivative of $V_E(t)$ along the trajectory of (18),one has

$$\dot{V}_{E1}(t)=\sum_{i=2}^{M}\hat{\eta}_i^T(t)\begin{bmatrix}R_E{}^{-1}\tilde{A}+\tilde{A}^TR_E{}^{-1} & -\lambda_{\sigma(t)}^iR_E{}^{-1}BK_3\\ * & 0\end{bmatrix}\hat{\eta}_i(t) \quad (19)$$

From the Assumption 3, one gets

$$\begin{aligned}\dot{V}_{E2}(t)&\le\eta^T(t)\left(I_{M-1}\otimes\Omega_E^{-1}\right)\eta(t)\\ &\quad-(1-\delta)\,\eta^T(t-\tau(t))\left(I_{M-1}\otimes\Omega_E^{-1}\right)\eta(t-\tau(t))\\ &=\sum_{i=2}^{M}\hat{\eta}_i^T(t)\begin{bmatrix}\Omega_E^{-1} & 0\\ 0 & -(1-\delta)\,\Omega_E^{-1}\end{bmatrix}\hat{\varphi}_i(t)\end{aligned} \quad (20)$$

Let $\varpi_i=\left[\tilde{A},-\lambda_{\sigma(t)}^iBK_3\right]$, based on Assumption 3 and Lemma 2, one has

$$\dot{V}_{E3}(t)\le\sum_{i=2}^{M}\hat{\eta}_i^T(t)\left(\varpi_i^TS_E^{-1}\varpi_i+\left(\begin{bmatrix}M_1^T+M_1 & -M_1^T+M_2\\ * & -M_2^T-M_2\end{bmatrix}+\sigma\begin{bmatrix}M_1^T\\ M_2^T\end{bmatrix}S_E^{-1}\left[M_1,M_2\right]\right)\hat{\eta}_i(t)\right. \quad (21)$$

Let $M_1=-R_E^{-1},M_2=\Omega_E^{-1}$, it can be obtained that

$$\dot{V}_E(t)\le\sum_{i=2}^{M}\hat{\eta}_i^T(t)\mathrm{Z}_i\hat{\eta}_i(t) \quad (22)$$

where

$$\begin{aligned}\mathrm{Z}_i&=\mathrm{T}_i+\sigma\varpi_i^TS_E^{-1}\varpi_i+\sigma\begin{bmatrix}-R_E^{-T}\\ \Omega_E^{-T}\end{bmatrix}S_E^{-1}\left[-R_E^{-1},\Omega_E^{-1}\right],\\ \mathrm{T}_i&=\begin{bmatrix}\mathrm{T}_{i11} & \Omega_E^{-1}+R_E^{-1}-\lambda_{\sigma(t)}^iR_E^{-1}BK_3\\ * & -(3-\delta)\Omega_E^{-1}\end{bmatrix},\\ \mathrm{T}_{i11}&=-2R_E^{-1}+R_E^{-1}\tilde{A}+\tilde{A}^TR_E^{-1}+\Omega_E^{-1}.\end{aligned}$$

Based Schur complement lemma [19], it can be concluded that $\mathrm{Z}_i<0$ is equivalent to $\Psi_i<0$

$$\Psi_i=\begin{bmatrix}T_i & \sigma\varpi_i^T & \sigma\begin{bmatrix}-R_E^{-1} & -\Omega_E^{-1}\end{bmatrix}\\ * & \sigma S_E^{-1} & 0\\ * & * & -\sigma S_E^{-1}\end{bmatrix}<0$$

Let $\Gamma=\begin{bmatrix}R_E & 0\\ \Omega_E & \Omega_E\end{bmatrix}$, and $\bar{\Gamma}=diag\left\{T,I,S\right\}$, one gets

$$\bar{\Gamma}^T\psi_i\bar{\Gamma}=\begin{bmatrix}\Gamma^TT\Gamma_i & \sigma\Gamma^T\varpi_i^T & \sigma\begin{bmatrix}0 & S_E\end{bmatrix}\\ * & \sigma S_E & 0\\ * & * & -\sigma S_E\end{bmatrix}$$

According to Schur complement lemma [19], $\prod(\bar{\lambda}_i^E)<0$ are equivalent to $\prod(\lambda_{\sigma(t)}^i)<0\,(i=2,3,\cdots,M,\sigma(t)=1,2,\cdots,p)$, from (18) to (22), one has $\lim_{t\to\infty}\varphi(t)=0$.

If (16) holds, it can get that,

$$\left(L_{\sigma(t)}^E\otimes(A+BK_1)\right)h_E(t)-\left(L_{\sigma(t)}^E\otimes I_n\right)\dot{h}_E(t)\equiv 0 \quad (23)$$

Substituting $L_{\sigma(t)}^E=Udiag(0,\tilde{U}^TL_{\sigma(t)}^E\tilde{U})U^T$ into (23) and then pre-multiplying the left and right of (23) by $U^T\otimes I_n$, it has

$$\begin{aligned}\left(\left(\tilde{U}^TL_{\sigma(t)}^E\tilde{U}\right)\otimes I_n\right)&\left[\left(U^T\otimes(A+BK_1)\right)h_E(t)\right.\\ &\left.-\left(U^T\otimes I_n\right)\dot{h}_E(t)\right]\equiv 0\end{aligned} \quad (24)$$

Since $\tilde{U}^TL_{\sigma(t)}^E\tilde{U}$ is invertible, it could pre-multiply the both sides of (24) by $\left(\tilde{U}^TL_{\sigma(t)}^E\tilde{U}\right)^{-1}\otimes I_n$, one gets

$$(\tilde{U}^T\otimes(A+BK_1))h(t)-(\tilde{U}^T\otimes I_n)\dot{h}(t)\equiv 0 \quad (25)$$

Therefore, based on Algorithm 1, The leaders in MAS (1) achieve time-varying formation specified by $h_E(t)$.

Step 2: The followers can converge to the convex hull spanned by the leaders.

Let $\tilde{\xi}_i(t)=\hat{\xi}_i(t)-z(t)$. It can be verified from (5) that

$$\begin{aligned}\dot{\tilde{\xi}}_i(t)&=(I_M\otimes(A+BK_3))\,\tilde{\xi}_i(t)-K_5\left[b_i^{\sigma(t)}\tilde{\xi}_i(t-\tau(t))\right.\\ &\quad\left.+\sum_{k=M+1}^{M+N}\omega_{ik}\left(\tilde{\xi}_i(t-\tau(t))-\tilde{\xi}_k(t-\tau(t))\right)\right]\\ &\quad+L_{\sigma(t)}^E\otimes BK_2\tilde{z}(t-\tau(t))\end{aligned} \quad (26)$$

Let $\varsigma(t)=\left[\tilde{\xi}_1^T(t),\tilde{\xi}_2^T(t),\cdots,\tilde{\xi}_N^T(t)\right]^T$, then (26) could be rewritten as

$$\begin{aligned}\dot{\varsigma}(t)&=(I_N\otimes\tilde{A})\varsigma(t)-(H_{\sigma(t)}\otimes K_5)\varsigma(t-\tau(t))\\ &\quad+I_N\otimes\left(L_{\sigma(t)}^E\otimes BK_2\tilde{z}(t-\tau(t))\right)\end{aligned} \quad (27)$$

where $H_{\sigma(t)} = B_{\sigma(t)} + L^F_{\sigma(t)}$. $B_{\sigma(t)}=diag\left[b_1^{\sigma(t)}, b_2^{\sigma(t)}, \cdots, b_N^{\sigma(t)}\right]$.

Consider the stability of system (27). Because $\lim\limits_{t\to 0}\left(L^E_{\sigma(t)}\otimes BK_2\tilde{z}\left(t-\tau\left(t\right)\right)\right)=0$, according to [16], the stability of system (27) is equivalent to (28).

$$\dot{\varsigma}(t) = (I_N\otimes\tilde{A})\varsigma(t) - (H_{\sigma(t)}\otimes K_5)\varsigma\left(t-\tau(t)\right) \quad (28)$$

By a similar analysis as for system (17), system (27) is asymptotically stable, which means that the state observer can estimate the state of leaders.

Let containment error $\tilde{x}_i(t) = x_i(t) - \sum\limits_{j=1}^{M}\rho_{ij}z_j(t)$. Based on the Alogrithm 1, $K_4=K_3-K_1$ and $\lim\limits_{t\to\infty}\left(\tilde{z}_k\left(t-\tau\left(t\right)\right)-\tilde{z}_j\left(t-\tau\left(t\right)\right)\right) = 0$, it follows form (7) and (9) that

$$\dot{\tilde{x}}_i(t) = (A+BK_3)\tilde{x}_i(t) - BK_4\sum_{j=1}^{M}\rho_{ij}\tilde{\xi}_{i,j}(t) \quad (29)$$

From Algorithm 1 and (28), the gain matrix K_3 is chose to make $A+BK_3$ Hurwitz. Since $\tilde{\xi}_i(t)=\left[\tilde{\xi}^T_{i,1}(t), \tilde{\xi}^T_{i,2}(t), \cdots, \tilde{\xi}^T_{i,M}(t)\right]^T$ implies $\lim\limits_{t\to\infty}\tilde{\xi}^T_{i,j}(t) = 0\,(j=1,2,\cdots,M)$. Therefore, $\lim\limits_{t\to\infty}\dot{\tilde{x}}_i(t) = 0$, which means MAS(1) can achieve formation-containment control under switching interaction topologies and time-varying delays. This completes the proof. ■

Remark 1: In [7] and [14], formation or containment control problem with both time-varying delays and switching topologies are studied respectively. From Theorem 1 and Algorithm 1, one sees that MAS can achieve formation-containment with both time delays and switching interaction topologies, which means the proposed approach in this paper is more versatile.

IV. Numerical simulations

In this section, a numerical simulation is given to demonstrate the effectiveness of theoretical results obtained by the previous section.

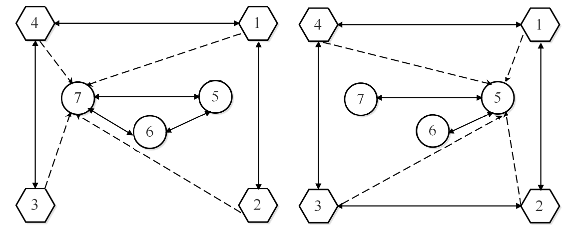
Consider a third-order MAS with four leaders and three followers, where the dynamics of each agent is presented by (1) with

$$A=\begin{bmatrix}-3 & 1 & -3\\ 0 & 0 & 1\\ -2 & -4 & -6\end{bmatrix}, B=\begin{bmatrix}0\\0\\1\end{bmatrix}$$

Suppose that $\tau(t) = 0.05+0.01\cos(t)$, Fig.1 shows the interaction topologies with 0-1 weight. Suppose that interaction topologies are randomly chosen form Fig. 1 with interval $T_d = 10s$.

The time-varying formation is specified as follows:

$$h_i(t) = \begin{bmatrix}15\sin\left(t+\frac{(i-1)\pi}{2}\right)\\ 15\cos\left(t+\frac{(i-1)\pi}{2}\right)\\ -15\sin\left(t+\frac{(i-1)\pi}{2}\right)\end{bmatrix} (i=1,2,3,4)$$



(a) G_1 (b) G_2

Fig. 1: Switching topologies

From $h_i(t)\,(i=1,2,3,4)$, it can be obtained that if the formation is achieved, the four leaders will locate at the four vertices of a square respectively, and keep rotation with an angular velocity of $1rad/s$.

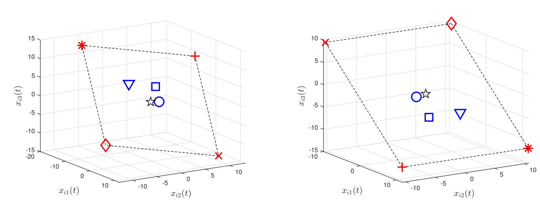
According to Algorithm 1, K_1 can be chosen as $\begin{bmatrix}-3 & 3 & 1\end{bmatrix}$ and the eigenvalues of $A+BK_1$ are be specified at -8, j and $-j$. Choose $K_3 = \begin{bmatrix}2 & 2 & 3\end{bmatrix}$ to place the eigenvalues of $A+BK_3$ at -1, -2 and -3. $K_4=\begin{bmatrix}5 & -1 & 2\end{bmatrix}$, using Step 2 and Step 5 in Algorithm 1, K_2 and K_5 can be given as follow:

$$K_2=\begin{bmatrix}-0.6995 & 0.2506 & 0.2901\end{bmatrix}$$

$$K_5=I_4\otimes\begin{bmatrix}0.1099 & 0.0819 & -0.3413\\ 0.0756 & 0.4368 & 0.0607\\ -0.3723 & 0.0165 & -0.1862\end{bmatrix}$$

The initial state vectors of four leaders and followers are described by $z_{ij}(0) = 3(\Theta-0.5)$ $(i=1,2,3,4; j=1,2,3)$; $x_{ij}(0) = 3(\Theta-0.5)$ $(i=5,6,7; j=1,2,3)$, where Θ is a pseudorandom value with a uniform distribution on the interval $(0,1)$.

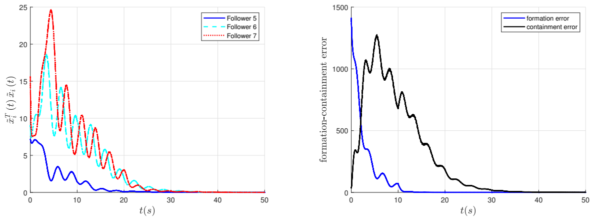
The desired state containment for third-order MAS is specified by $\rho_{51}=\frac{1}{6}$, $\rho_{52}=\frac{1}{3}$, $\rho_{53}=\frac{1}{4}$, $\rho_{54}=\frac{1}{4}$, $\rho_{61}=\frac{1}{3}$, $\rho_{62}=\frac{1}{3}$, $\rho_{63}=\frac{1}{4}$, $\rho_{64}=\frac{1}{12}$, $\rho_{71}=\frac{1}{2}$, $\rho_{62}=\frac{1}{6}$, $\rho_{63}=\frac{1}{6}$, $\rho_{64}=\frac{1}{6}$.



(a) $t=36s$ (b) $t=48s$

Fig. 2: Snapshots of seven agents $(t=36s; t=48s)$

Fig. 2 shows the state snapshots of seven agents, where the state of leaders and followers are denoted by different colors. The black pentagram denotes the state of the formation reference. Fig. 3(a) denotes all the followers can acquire the leaders' state. Fig. 3(b) displays the time-varying formation error and containment error within $t=50s$. Form Fig. 2-3, one sees that MAS with time-varying delays and switching interaction topologies achieves the desired formation-containment.



(a) $state\ observers'\ errors$ (b) $system\ errors$

Fig. 3: Curves of formation-containment control errors

V. CONCLUSIONS

Formation-containment control problems for general linear MASs with time-varying delays and switching interaction topologies were studied. Based on consensus approaches, a formation-containment protocol was proposed. Then, an Algorithm was presented to obtain the constant matrices in the protocol. Using LMI technique and Lyapunov-krasovskii stability theory, the stability could be proved. Simulation results showed that the practical results were effective for formation-containment problems with both time-varying delays and switching interaction topologies.

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