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Brief Paper

Distributed adaptive fixed-time formation tracking for heterogeneous multi-agent systems ☆

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ABSTRACT

This article investigates the problem of fixed-time time-varying formation tracking control (FTC) for heterogeneous linear multiagent systems (MASs) under the directed communication graph. It is assumed that the Laplacian matrix associated with the communication graph is unavailable and that the system matrices of the leader are only available to its neighboring followers. This differs from many existing works on fixed-time FTC problems where the communication graphs are typically undirected and protocol designs often rely on certain global information. A novel distributed observer is first put forward to estimate both the state and system matrices of the leader in fixed time. Then, an adaptive scheme is developed to solve the time-varying regulator equations resulting from the estimated leader system matrices in fixed time. Based on the proposed observer and the adaptive solutions to the regulator equations, a distributed adaptive fixed-time FTC protocol is further proposed via coordinate transformation techniques. It is shown that our proposed controllers do not require the input matrices of the followers to be of full row rank. It is also shown that the concerned fixed-time FTC problem can be solved with the proposed fixed-time FTC strategy in a distributed manner. Our results can be directly applied to solve both the adaptive fixed-time cooperative output regulation problem and the leader-following consensus problems of MASs under the directed graph. Finally, the effectiveness of the proposed fixed-time FTC strategy is demonstrated through a numerical example.

1. Introduction

Cooperative control of multi-agent systems (MASs) has attracted considerable attention over the recent decades from a variety of fields, such as robotics, sensor networks, and power systems (Cai et al., 2017; Fang & Wen, 2025; Fax & Murray, 2004; Firouzbahrami & Nobakhti, 2022; He et al., 2025, 2024; Lin et al., 2022; Olfati-Saber & Murray, 2004; Sun et al., 2021; Wieland et al., 2011). A prominent research topic of cooperative control is formation control, which aims to establish control protocols to achieve a specific formation (Huang et al., 2024; Liu & Li, 2024; Ren, 2007; Yang et al., 2023). In addition to simply forming a formation, in many real-world applications, MASs are also required to follow a trajectory provided by a leader. This requirement gives rise to the so-called formation tracking control (FTC) problem. Various FTC strategies have been developed for different types of MASs, including first or second-order agent dynamics, general homogeneous agent dynamics, and general heterogeneous agent dynamics (Feng et al., 2024; Li et al., 2024; Liu et al., 2022; Porfiri et al., 2007).

Most of the above-mentioned results on cooperative control of MASs focus on guaranteeing either asymptotic or exponential stability of the closed-loop systems. However, in many practical applications, it is often necessary or desirable to achieve those objectives in a finite time. Known for its fast convergence, high precision, and strong robustness, finite-time control has attracted significant attention and numerous notable results have been recently reported, see, for example, Cai et al. (2020), Fu and Wang (2016), Wang et al. (2024, 2020, 2021), Xiao et al. (2009). Unfortunately, one notable limitation of finite-time control is that the upper bound of the settling time depends on the initial conditions of the concerned MASs. To address this issue, a fixed-time control protocol, whose upper bound of the settling time does not depend on the initial conditions, was first proposed in Polyakov (2011). Since then, extensive research has been conducted on cooperative control of MASs with fixed-time convergence (Cheng et al., 2022; Dong & Chen, 2022; Du et al., 2020; Zuo et al., 2017).

It is worth noting that while the distributed control protocols of MASs in the above-mentioned literature use only local

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information from neighboring agents to achieve fixed-time cooperative control, their protocol designs often rely on certain global information of MASs, for instance, the eigenvalues of the matrices associated with the communication graphs, or the system matrices of the exosystem. To address these issues, adaptive techniques have been adopted for distributed fixed-time control protocols, see, e.g., Cai et al. (2022), Jiang et al. (2023), Zhang et al. (2021), Zuo et al. (2023). However, the adaptive fixed-time control protocols in Jiang et al. (2023), Zhang et al. (2021), Zuo et al. (2023) depend on the assumption that the communication graphs are either undirected or strongly connected. Moreover, the designs of those protocols in Cai et al. (2022), Jiang et al. (2023), Zhang et al. (2021) require a restrictive assumption, that is, the input matrices of the followers are of full-row rank, which might not be satisfied in many engineering applications. In addition, it was not shown in those works that the solutions to the regulator equations can be obtained within a fixed time. To the best of our knowledge, the problem of fixed-time FTC for heterogeneous MASs under the directed communication graph, which does not require the global information associated with the graph nor the system matrices of the leader, is yet to be addressed, thereby motivating this work.

This work investigates the problem of fully distributed adaptive fixed-time FTC for heterogeneous MASs under the directed graph. The main contributions of this work are summarized as follows. First, a novel distributed adaptive fixed-time observer is proposed under the directed communication graph to estimate the state and the system matrices of the leader without requiring any global information. This contrasts with the fixed-time observers presented in previous studies (Cheng et al., 2022; Dong & Chen, 2022; Du et al., 2020; Zuo et al., 2017), which are not fully distributed. Moreover, the asymmetry of the Laplacian matrix in the directed graph presents additional challenges for both fully distributed observer design and stability analysis compared to their counterparts in undirected or strongly connected graphs (Jiang et al., 2023; Zhang et al., 2021; Zuo et al., 2023). Second, compared to the approaches in Cai et al. (2022), Jiang et al. (2023), Zhang et al. (2021), where the solutions to the regulator equations can only be obtained when time goes to infinity, a novel adaptive scheme is developed to solve the regulator equations in fixed time, and also in a fully distributed manner.

Notation: I_n represent the n-dimensional identity matrix. $\mathbf{0}$ denotes the zero matrix, and its dimension can be known from the context. The Kronecker product is denoted as $\otimes$. Let $\mathrm{diag}\{a_1, \ldots, a_n\}$ be the diagonal matrix with a_i, $i = 1, \ldots, n$, as its diagonal entries. $\|\cdot\|$ and $\|\cdot\|_p$ denote the Euclidean norm and p-norm for vectors, respectively. For a vector $\mathcal{X} = [x_1, \ldots, x_n]^\top \in \mathbb{R}^n$, define $\mathrm{sign}^\alpha(\mathcal{X}) = [\mathrm{sign}(x_1)|x_1|^\alpha, \ldots, \mathrm{sign}(x_n)|x_n|^\alpha]^\top$, where $\alpha > 0$ and $\mathrm{sign}(\cdot)$ is the sign function. Let $\min(\cdot)$ and $\max(\cdot)$ be the minimum and maximum element of an array, respectively. Consider a matrix $\mathcal{A} = [a_1, \ldots, a_q] \in \mathbb{R}^{p\times q}$, where $a_i \in \mathbb{R}^{p\times 1}$, $i = 1, \ldots, q$. Define $\mathcal{A}_{\mathrm{vec}} = \mathrm{vec}(\mathcal{A}) = [a_1^\top, \ldots, a_q^\top]^\top$, and let $\mathcal{A} = \mathrm{M}_p^q(\mathcal{A}_{\mathrm{vec}})$ represent the transformation between the matrix $\mathcal{A}$ and the corresponding column vector $\mathcal{A}_{\mathrm{vec}}$.

2. Preliminaries and problem formulation

2.1. Algebraic graph theory

Considers a heterogeneous MAS consisting of M followers and one leader. An agent is classified as a leader if it has no neighbors; otherwise, it is defined as a follower. The digraph among the followers is defined as $\mathcal{G}_f = (\mathcal{V}_f, \mathcal{E}_f, \mathcal{A}_f)$, which consists of $\mathcal{V}_f = \{1, \ldots, M\}$, $\mathcal{E}_f \subseteq \mathcal{V}_f \times \mathcal{V}_f$, and the adjacency matrix $\mathcal{A}_f = [a_{ij}]_{M\times M}$ with $a_{ij} > 0 \Leftrightarrow (j, i) \in \mathcal{E}_f$ and $a_{ij} = 0$ otherwise. Let a_{i0} be the pinning gain from the leader to the ith follower. In particular, $a_{i0} > 0$ if information transmission between them is feasible; otherwise, $a_{i0} = 0$. Let $\bar{\mathcal{G}} = (\bar{\mathcal{V}}, \bar{\mathcal{E}}, \bar{\mathcal{A}})$ be the directed communication topology among the followers and the leader. A spanning tree is a directed graph where at least one root node is connected by a directed path to every other node.

2.2. Problem formulation

The dynamics of the ith follower are described by

$$\dot{x}_i(t) = A_i x_i(t) + B_i u_i(t),$$
$$y_i(t) = C_i x_i(t), \ i \in \mathcal{V}_f, \tag{1}$$

where $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$, $y_i \in \mathbb{R}^p$ represent the state, control input, and output of the ith follower, respectively, A_i, B_i, and C_i are the system matrices satisfying $\mathrm{rank}(B_i) = m_i$.

The dynamics of the leader indexed by 0 are given as

$$\dot{x}_0(t) = S x_0(t),$$
$$y_0(t) = F x_0(t), \tag{2}$$

where $x_0 \in \mathbb{R}^{n_0}$ and $y_0 \in \mathbb{R}^p$ represent the state and output of the leader, respectively, S and F are system matrices of the leader.

The piecewise continuous differentiable formation vector $h_{oi}(t) \in \mathbb{R}^p$ of the ith follower is generated by the following system,

$$\dot{h}_i(t) = S_h h_i(t),$$
$$h_{oi}(t) = F_h h_i(t), \ i \in \mathcal{V}_f, \tag{3}$$

where $h_i \in \mathbb{R}^l$, S_h and F_h are matrices of compatible dimensions.

Then, the fixed-time FTC problem under consideration is defined as follows.

Definition 1 (*Fixed-time FTC problem*)**.** Given the heterogeneous MAS consisting of (1) and (2) under the directed graph $\bar{\mathcal{G}}$, design a distributed control protocol for each follower such that

$$\lim_{t\to T_o} (y_i(t) - h_{oi}(t) - y_0(t)) = 0,$$
$$y_i(t) - h_{oi}(t) - y_0(t) = 0, \ t \geq T_o, \ \forall i \in \mathcal{V}_f, \tag{4}$$

where $T_o > 0$ represents the settling time that is independent of the initial conditions.

This paper aims to design a fully distributed fixed-time control protocol so that the fixed-time FTC problem can be solved in a fully distributed manner without global information, such as the eigenvalues of the matrices associated with the communication graphs, or the system matrices of the leader.

Remark 1. The formation vector $h_{oi}(t)$, generated by Eq. (3), specifies the desired relative offset of $y_i(t)$ relative to $y_0(t)$. Various formation shapes can be generated using Eq. (3). For instance, setting $S_h = \mathbf{0}$ allows (3) to specify a non-rotating time-invariant formation shape. Additionally, a three-dimensional circular formation, as discussed in Li et al. (2024), Wang et al. (2024), can be formed by setting $S_h = \begin{pmatrix} 0 & 1 \\ -c^2 & 0 \end{pmatrix}$ and $F_h = \begin{pmatrix} 1 & 0 \end{pmatrix}$ for the X, Y, and Z axes, respectively, where c denotes a positive constant.

Before presenting the main results, the following assumptions and lemma are put forward.

Assumption 1. The digraph $\bar{\mathcal{G}}$ contains a spanning tree with the leader as its root node.

Then, the corresponding Laplacian matrix $\bar{\mathcal{L}}$ can be partitioned as $\begin{bmatrix} \mathcal{L}_f & \mathcal{L}_l \\ \mathbf{0} & 0 \end{bmatrix}$, where $\mathcal{L}_l = [-a_{0i}]_{M\times 1}$, $\mathcal{L}_f = [l_{ij}]_{M\times M}$ with $l_{ij} = -a_{ij}$ for $i \neq j$, and $l_{ii} = \sum_{k=1}^{M} a_{ik} + a_{i0}$.

Lemma 1 (Qu, 2009). *Under Assumption 1, all eigenvalues of $\mathcal{L}_f$ have positive real parts. Moreover, there exists a real diagonal matrix $\bar{\mathcal{D}}$ such that $\bar{\mathcal{Q}} = \bar{\mathcal{D}}\mathcal{L}_f + \mathcal{L}_f^\top \bar{\mathcal{D}} > 0$.*

Define $\mathcal{D} = \text{diag}\{d_1, \ldots, d_M\} = \frac{2\bar{\mathcal{D}}}{\lambda_{\min}}$, where $\lambda_{\min} > 0$ be the smallest eigenvalue of $\bar{\mathcal{Q}}$, $d_i > 0$, $i = 1, \ldots, M$. Then, by Lemma 1, one has $\mathcal{Q} = \mathcal{D}\mathcal{L}_f + \mathcal{L}_f^\top \mathcal{D} \geq 2\mathrm{I}_M$.

Assumption 2. For any $i \in \mathcal{V}_f$, (A_i, B_i) are controllable.

Assumption 3. The regulator equations,

$$X_i S = A_i X_i + B_i U_i, \tag{5a}$$

$$0 = C_i X_i - F, \tag{5b}$$

have solution pairs (X_i, U_i) for all $i \in \mathcal{V}_f$.

Assumption 4. The linear matrix equations,

$$X_{hi} S_h = A_i X_{hi} + B_i U_{hi}, \tag{6a}$$

$$0 = C_i X_{hi} - F_h, \tag{6b}$$

have solution pairs (X_{hi}, U_{hi}) for all $i \in \mathcal{V}_f$.

Remark 2. Assumptions 1–4 are *necessary* for achieving the FTC for heterogeneous MASs as in Cai et al. (2022), Cheng et al. (2023). The solvability of Eqs. (5) can be referred to Huang (2004).

3. Main results

In this section, fully distributed fixed-time observers, the adaptive scheme to solve the regulator equations, and distributed fixed-time control protocols will be provided to address the fixed-time FTC problem for the concerned MAS.

3.1. Distributed adaptive fixed-time observers

To estimate the state and the system matrix of the leader, a fixed-time distributed observer is first proposed for each follower as follows:

$$\dot{\eta}_i = S_i \eta_i - (c_{1i} + \varepsilon_i^\top \varepsilon_i)\varepsilon_i - c_2 \text{sign}^{\alpha_1}(\varepsilon_i) - c_3 \text{sign}^{\alpha_2}(\varepsilon_i), \tag{7a}$$

$$\dot{c}_{1i} = \varepsilon_i^\top \varepsilon_i, \tag{7b}$$

$$\dot{\hat{S}}_{0i} = -c_4 \text{sign}^{\beta_1}(\bar{S}_{0i}) - c_5 \text{sign}^{\beta_2}(\bar{S}_{0i}), \tag{7c}$$

$$\dot{\hat{F}}_{0i} = -c_6 \text{sign}^{\gamma_1}(\bar{F}_{0i}) - c_7 \text{sign}^{\gamma_2}(\bar{F}_{0i}), \tag{7d}$$

where $\varepsilon_i = \sum_{j=1}^M a_{ij}(\eta_i - \eta_j) + a_{i0}(\eta_i - x_0)$, $\bar{S}_{0i} = \sum_{j=1}^M a_{ij}(\hat{S}_{0i} - \hat{S}_{0j}) + a_{i0}(\hat{S}_{0i} - S_0)$, $\bar{F}_{0i} = \sum_{j=1}^M a_{ij}(\hat{F}_{0i} - \hat{F}_{0j}) + a_{i0}(\hat{F}_{0i} - F_0)$, η_i represents the state of the distributed observer used to estimate x_0, $S_0 = \text{vec}(S)$, $F_0 = \text{vec}(F)$, $\bar{S}_{0i}$, $\bar{F}_{0i}$, and ε_i denote the neighboring relative estimation errors for S_0, F_0, and x_0, respectively, $\hat{S}_{0i} \in \mathbb{R}^{n_0^2}$ and $\hat{F}_{0i} \in \mathbb{R}^{pn_0}$ are the estimates of the vector S_0 and F_0 respectively via follower i, $S_i = \mathrm{M}_{n_0}^{n_0}(\hat{S}_{0i})$, $F_i = \mathrm{M}_p^{n_0}(\hat{F}_{0i})$, c_{1i} is the adaptive updating gain satisfying $c_{1i}(0) > 0$, $c_j, j = 2, \ldots, 6$, α_1, α_2, β_1, β_2, γ_1, and γ_2 are positive constants to be determined later.

The following theorem shows that the observer errors $\tilde{S}_{0i} \triangleq \hat{S}_{0i} - S_0$, $\tilde{F}_{0i} \triangleq \hat{F}_{0i} - F_0$, and $\tilde{\eta}_i \triangleq \eta_i(t) - x_0(t)$ are fixed-time convergent.

Theorem 1. *Consider the leader (2) and the distributed observers (7a)–(7d). Select $c_i > 0$, $i = 2, \ldots, 6$, $0 < \alpha_1 < 1$, $\alpha_2 > \frac{1}{\alpha_1} > 1$, $0 < \beta_1 < 1$, $\beta_2 > \frac{1}{\beta_1} > 1$, $0 < \gamma_1 < 1$, and $\gamma_2 > \frac{1}{\gamma_1} > 1$. Then, for all $i \in \mathcal{V}_f$, the following properties hold:*

1. *$\lim_{t \to T_S} \tilde{S}_{0i}(t) = 0$ and $\tilde{S}_{0i}(t) = 0$, $t \geq T_S$,*
2. *$\lim_{t \to T_F} \tilde{F}_{0i}(t) = 0$ and $\tilde{F}_{0i}(t) = 0$, $t \geq T_F$,*
3. *$\lim_{t \to T_\eta} \tilde{\eta}_i(t) = 0$ and $\tilde{\eta}_i(t) = 0$, $t \geq T_\eta$,*

where $T_S \geq 0$, $T_F \geq 0$, and $T_\eta \geq 0$ represent the settling times regardless of any initial conditions.

Proof. The proof of Theorem 1 includes three parts. Firstly, we show the fixed-time convergence of $\tilde{S}_{0i}$ and $\tilde{F}_{0i}$. Then, we establish the boundedness of the observer error $\tilde{\eta}_i$ in $[0, T_S)$, ensuring that no finite-time escape occurs. Lastly, we show that the observer error $\tilde{\eta}_i$ converges to zero in fixed time.

Part (i): Fixed-time convergence of $\tilde{S}_{0i}$ and $\tilde{F}_{0i}$.

Not that $\tilde{S}_{0i} \triangleq \hat{S}_{0i} - S_0$. Letting $\tilde{S} = [\tilde{S}_{01}^\top, \ldots, \tilde{S}_{0M}^\top]^\top$, it then follows from (7c) that

$$\dot{\tilde{S}} = -\breve{S}, \tag{8}$$

where $\breve{S} = [\breve{S}_{01}^\top, \ldots, \breve{S}_{0M}^\top]^\top$ with $\breve{S}_{0i} = c_4 \text{sign}^{\beta_1}(\bar{S}_{0i}) + c_5 \text{sign}^{\beta_2}(\bar{S}_{0i})$.

Let $\bar{S} = [\bar{S}_{01}^\top, \ldots, \bar{S}_{0M}^\top]^\top$. Noting that $\bar{S} = (\mathcal{L}_f \otimes \mathrm{I}_{n_0^2})\tilde{S}$, by (8), the dynamics of $\bar{S}$ can be given by

$$\dot{\bar{S}} = -(\mathcal{L}_f \otimes \mathrm{I}_{n_0^2})\breve{S}. \tag{9}$$

Consider the following Lyapunov function candidate,

$$V_1 = \sum_{i=1}^M \left(\frac{c_4 d_i}{1+\beta_1} \|\bar{S}_{0i}\|_{1+\beta_1}^{1+\beta_1} + \frac{c_5 d_i}{1+\beta_2} \|\bar{S}_{0i}\|_{1+\beta_2}^{1+\beta_2} \right). \tag{10}$$

The time derivative of V_1 along the trajectory of the system (9) is calculated as

$$\dot{V}_1 = -\frac{1}{2}\breve{S}^\top \left((\mathcal{L}_f^\top \mathcal{D} + \mathcal{D}\mathcal{L}_f) \otimes \mathrm{I}_{n_0^2} \right) \breve{S} = -\|\breve{S}\|^2. \tag{11}$$

Given that $c_4 \text{sign}^{\beta_1}(\bar{S}_{0i})$ and $c_5 \text{sign}^{\beta_2}(\bar{S}_{0i})$ exhibit component-wise sign consistency across all their respective elements, the terms in the expansion of $\|\breve{S}_{0i}\|^2$, including both square terms and cross terms, are non-negative. Then, one has $\|\breve{S}\|^2 \geq \sum_{i=1}^M c_4^2 \|\bar{S}_{0i}\|_{2\beta_1}^{2\beta_1} + \sum_{i=1}^M c_5^2 \|\bar{S}_{0i}\|_{2\beta_2}^{2\beta_2}$. Since $0 < \beta_1 < 1$ and $\beta_2 > 1$, via Lemma 3 given in Appendix A, one has $\sum_{i=1}^M \|\bar{S}_{0i}\|_{2\beta_1}^{2\beta_1} \geq \left(\|\bar{S}\|^2\right)^{\beta_1}$ and $\sum_{i=1}^M \|\bar{S}_{0i}\|_{2\beta_2}^{2\beta_2} \geq \frac{1}{(n_0^2 M)^{\beta_2 - 1}} \left(\|\bar{S}\|^2\right)^{\beta_2}$. It then follows that

$$\dot{V}_1 \leq -k_1 \left(\left(\|\bar{S}\|^2\right)^{\beta_1} + \left(\|\bar{S}\|^2\right)^{\beta_2} \right), \tag{12}$$

where $k_1 = \min\{c_4^2, \frac{c_5^2}{(n_0^2 M)^{\beta_2 - 1}}\} > 0$.

Since $0 < \frac{1+\beta_1}{2} < 1$ and $\frac{1+\beta_2}{2} > 1$, one has $\sum_{i=1}^M \frac{c_4 d_i}{1+\beta_1} \|\bar{S}_i\|_{1+\beta_1}^{1+\beta_1} \leq \bar{c}_4 \left(\|\bar{S}\|^2\right)^{\frac{1+\beta_1}{2}}$ and $\sum_{i=1}^M \frac{c_5 d_i}{1+\beta_2} \|\bar{S}_i\|_{1+\beta_2}^{1+\beta_2} \leq \bar{c}_5 \left(\|\bar{S}\|^2\right)^{\frac{1+\beta_2}{2}}$, where $\bar{c}_4 = (\frac{c_4 d_{\max}}{1+\beta_1})(n_0^2 M)^{\frac{1-\beta_1}{2}}$, $\bar{c}_5 = \frac{c_5 d_{\max}}{1+\beta_2}$, and $d_{\max} = \max\{d_1, \ldots, d_M\}$. Since $0 < \frac{2\beta_1}{1+\beta_1} < 1$ and $\frac{2\beta_2}{1+\beta_2} > 1$, it then follows that

$$V_1^{\frac{2\beta_1}{1+\beta_1}} \leq k_2 \left((\|\bar{S}\|^2)^{\beta_1} + (\|\bar{S}\|^2)^{\frac{\beta_1(1+\beta_2)}{1+\beta_1}} \right), \tag{13}$$

$$V_1^{\frac{2\beta_2}{1+\beta_2}} \leq k_3 \left((\|\bar{S}\|^2)^{\frac{\beta_2(1+\beta_1)}{1+\beta_2}} + (\|\bar{S}\|^2)^{\beta_2} \right), \tag{14}$$

where $k_2 = \max\{\bar{c}_4^{\frac{2\beta_1}{1+\beta_1}}, \bar{c}_5^{\frac{2\beta_1}{1+\beta_1}}\}$ and $k_3 = \max\{\bar{c}_4^{\frac{2\beta_2}{1+\beta_2}}, \bar{c}_5^{\frac{2\beta_2}{1+\beta_2}}\} \times 2^{\frac{\beta_2 - 1}{\beta_2 + 1}}$. Noting that $\beta_1 < \frac{\beta_1(1+\beta_2)}{1+\beta_1} < \beta_2$ and $\beta_1 < \frac{\beta_2(1+\beta_1)}{1+\beta_2} < \beta_2$, one has $(\|\bar{S}\|^2)^{\frac{\beta_1(1+\beta_2)}{1+\beta_1}} \leq (\|\bar{S}\|^2)^{\beta_1} + (\|\bar{S}\|^2)^{\beta_2}$ and $(\|\bar{S}\|^2)^{\frac{\beta_2(1+\beta_1)}{1+\beta_2}} \leq (\|\bar{S}\|^2)^{\beta_1} + (\|\bar{S}\|^2)^{\beta_2}$. It then follows from (13) and (14) that

$$V_1^{\frac{2\beta_1}{1+\beta_1}} \leq k_2 \left(2(\|\bar{S}\|^2)^{\beta_1} + (\|\bar{S}\|^2)^{\beta_2} \right), \tag{15}$$

$$V_1^{\frac{2\beta_2}{1+\beta_2}} \leq k_3 \left((\|\bar{S}\|^2)^{\beta_1} + 2(\|\bar{S}\|^2)^{\beta_2} \right). \tag{16}$$

The combination of (15) and (16) yields

$$\frac{1}{3k_2}V_1^{\frac{2\beta_1}{1+\beta_1}}+\frac{1}{3k_3}V_1^{\frac{2\beta_2}{1+\beta_2}}\leq(\|\bar{S}\|^2)^{\beta_1}+(\|\bar{S}\|^2)^{\beta_2}. \tag{17}$$

Then, one can obtain from (12) and (17) that

$$\dot{V}_1\leq-\frac{k_1}{3k_2}V_1^{\frac{2\beta_1}{1+\beta_1}}-\frac{k_1}{3k_3}V_1^{\frac{2\beta_2}{1+\beta_2}}. \tag{18}$$

Via Lemma 4 in Appendix A, one can obtain that system (9) is fixed-time stable with the settling time $T_S\leq\frac{3k_3(\beta_1+1)}{k_1(1-\beta_1)}+\frac{3k_2(\beta_2+1)}{k_1(\beta_2-1)}$. Moreover, since $\bar{S}=(\mathcal{L}_f\otimes \mathrm{I}_{n_0^2})\tilde{S}$ and $\mathcal{L}_f$ is nonsingular, one has $\lim_{t\to T_S}\tilde{S}_{0i}(t)=0$ and $\tilde{S}_{0i}(t)=0$ for $t\geq T_S$.

Letting $\tilde{F}=[\tilde{F}_{01}^\top,\dots,\tilde{F}_{0M}^\top]^\top$ and $\bar{F}=[\bar{F}_{01}^\top,\dots,\bar{F}_{0M}^\top]^\top$, one has $\bar{F}=(\mathcal{L}_f\otimes \mathrm{I}_{pn_0})\tilde{F}$. Similarly, by using the same approach in analyzing the fixed-time convergence of $\tilde{S}_{0i}(t)$, one can conclude that there exists a settling time $T_F\geq 0$ regardless of any initial states, such that $\lim_{t\to T_F}\tilde{F}_i(t)=0$ and $\tilde{F}_i(t)=0$ for $t\geq T_F$.

Part (ii): Boundedness of η_i in $t\in[0,T_S)$.

Note that $\tilde{\eta}_i=\eta_i-x_0$. From (2) and (7), one has

$$\dot{\tilde{\eta}}_i=S\tilde{\eta}_i-\bar{\varepsilon}_i+\tilde{S}_i\tilde{\eta}_i+\tilde{S}_ix_0, \tag{19}$$

where $\bar{\varepsilon}_i=\varphi_i\varepsilon_i+c_2\mathrm{sign}^{\alpha_1}(\varepsilon_i)+c_3\mathrm{sign}^{\alpha_2}(\varepsilon_i)$, $\varphi_i=c_{1i}+\varepsilon_i^\top\varepsilon_i$, and $\tilde{S}_i=\mathrm{M}_{n_0}^{n_0}(\tilde{S}_{0i})$.

Let $\tilde{\eta}=[\tilde{\eta}_1^\top,\dots,\tilde{\eta}_M^\top]^\top$, $\varepsilon=[\varepsilon_1^\top,\dots,\varepsilon_M^\top]^\top$, and $\bar{\varepsilon}=\left[\bar{\varepsilon}_1^\top,\dots,\bar{\varepsilon}_M^\top\right]^\top$. Noting that $\varepsilon=(\mathcal{L}_f\otimes \mathrm{I}_{n_0})\tilde{\eta}$, via (19), one has

$$\begin{aligned}\dot{\varepsilon}&=(I_M\otimes S)\,\varepsilon-\left(\mathcal{L}_f\otimes \mathrm{I}_{n_0}\right)\bar{\varepsilon}\\&\quad+(\mathcal{L}_f\otimes \mathrm{I}_{n_0})\tilde{S}^d(\mathcal{L}_f^{-1}\otimes \mathrm{I}_{n_0})\varepsilon+(\mathcal{L}_f\otimes \mathrm{I}_{n_0})\tilde{S}^d\bar{x}_0,\end{aligned} \tag{20}$$

where $\tilde{S}^d=\mathrm{diag}\{\tilde{S}_1,\dots,\tilde{S}_M\}$ and $\bar{x}_0=\mathbf{1}_M\otimes x_0$.

Consider the following Lyapunov function candidate,

$$V_2=V_{21}+V_{22}, \tag{21}$$

where $V_{21}=\sum_{i=1}^M\left(\frac{c_2d_i}{1+\alpha_1}\|\varepsilon_i\|_{1+\alpha_1}^{1+\alpha_1}+\frac{c_3d_i}{1+\alpha_2}\|\varepsilon_i\|_{1+\alpha_2}^{1+\alpha_2}\right)+\frac{1}{4}\sum_{i=1}^M d_i(\varphi_i+c_{1i})\varepsilon_i^\top\varepsilon_i$, $V_{22}=\frac{1}{4}\sum_{i=1}^M d_i(c_{1i}-\mu)^2$, and μ represents a positive constant to be specified.

The time derivative of V_2 along the trajectories of (20) satisfies

$$\begin{aligned}\dot{V}_2&=\bar{\varepsilon}^\top(\mathcal{D}\otimes S)\varepsilon-\frac{1}{2}\bar{\varepsilon}^\top\left(\mathcal{Q}\otimes \mathrm{I}_{n_0}\right)\bar{\varepsilon}+\frac{1}{2}\sum_{i=1}^M d_i(\varphi_i-\mu)\dot{c}_{1i}\\&\quad+\bar{\varepsilon}^\top(\mathcal{D}\mathcal{L}_f\otimes \mathrm{I}_{n_0})\tilde{S}^d(\mathcal{L}_F^{-1}\otimes \mathrm{I}_{n_0})\varepsilon+\bar{\varepsilon}^\top(\mathcal{D}\mathcal{L}_f\otimes \mathrm{I}_{n_0})\tilde{S}^d\bar{x}_0.\end{aligned} \tag{22}$$

Since $\mathcal{Q}=\mathcal{D}\mathcal{L}_f+\mathcal{L}_f^\top\mathcal{D}\geq 2\mathrm{I}_M$, via Young's inequality, one has

$$\bar{\varepsilon}^\top(\mathcal{D}\otimes S)\varepsilon-\frac{1}{2}\bar{\varepsilon}^\top\left(\mathcal{Q}\otimes \mathrm{I}_{n_0}\right)\bar{\varepsilon}\leq\frac{1}{2}(\|\mathcal{D}\otimes S\|^2\|\varepsilon\|^2-\|\bar{\varepsilon}\|^2). \tag{23}$$

From the result of part (i) and (2), it can be seen that $\tilde{S}_i$ and $\bar{x}_0$ are bounded for $t\in[0,T_S)$. Then, there exist positive constants ϖ_1 and ϖ_2 such that $\|(\mathcal{L}_f\otimes \mathrm{I}_{n_0})\tilde{S}^d(\mathcal{L}_f^{-1}\otimes \mathrm{I}_{n_0})\|^2\leq\varpi_1$ and $\|(\mathcal{L}_f\otimes \mathrm{I}_{n_0})\tilde{S}^d\bar{x}_0\|^2\leq\varpi_2$ hold for $t\in[0,T_S)$. Via Young's inequality, one has

$$\begin{aligned}&\bar{\varepsilon}^\top(\mathcal{L}_f\otimes \mathrm{I}_{n_0})\tilde{S}^d(\mathcal{L}_F^{-1}\otimes \mathrm{I}_{n_0})\varepsilon+\bar{\varepsilon}^\top(\mathcal{L}_f\otimes \mathrm{I}_{n_0})\tilde{S}^d\bar{x}_0\\&\leq\frac{1}{2}\|\bar{\varepsilon}\|^2+\varpi_1\|\varepsilon\|^2+\varpi_2.\end{aligned} \tag{24}$$

Since $0<\alpha_1<1$ and $\alpha_2>1$, via Lemma 3, one has $\sum_{i=1}^M\frac{c_2d_i}{1+\alpha_1}\|\varepsilon_i\|_{1+\alpha_1}^{1+\alpha_1}\geq\check{c}_2(\|\varepsilon\|^2)^{\frac{1+\alpha_1}{2}}$ and $\sum_{i=1}^M\frac{c_3d_i}{1+\alpha_2}\|\varepsilon_i\|_{1+\alpha_2}^{1+\alpha_2}\geq\check{c}_3(\|\varepsilon\|^2)^{\frac{1+\alpha_2}{2}}$, where $\check{c}_2=\frac{c_2d_{\min}}{1+\alpha_1}$, $\check{c}_3=\frac{c_3d_{\min}}{1+\alpha_2}(n_0M)^{\frac{1-\alpha_2}{2}}$, and $d_{\min}=\min\{d_1,\dots,d_M\}$. Noting that $\frac{1+\alpha_1}{2}<1<\frac{1+\alpha_2}{2}$, one has

$$V_{21}\geq k_{\check{c}}\left((\|\varepsilon\|^2)^{\frac{1+\alpha_1}{2}}+(\|\varepsilon\|^2)^{\frac{1+\alpha_2}{2}}\right)\geq k_{\check{c}}\|\varepsilon\|^2, \tag{25}$$

where $k_{\check{c}}=\min\{\check{c}_2,\check{c}_3\}$.

Moreover, via (7b) and (21), one can obtain $\frac{1}{2}\sum_{i=1}^M d_i(\varphi_i-\mu)\dot{c}_{1i}\leq 2V_{21}$. Then, it follows from (22)–(25) that

$$\dot{V}_2\leq k_\varepsilon V_{21}+\bar{\varpi}_2\leq k_\varepsilon V_2+\bar{\varpi}_2, \tag{26}$$

where $k_\varepsilon=\frac{\|\mathcal{D}\otimes S\|^2+2\bar{\varpi}_1}{2k_{\check{c}}}+2$.

One can further conclude from (26) that $V_2(t)$ is bounded in $t\in[0,T_S)$, which implies that ε_i and η_i are bounded in $[0,T_S)$.

Part (iii): Fixed-time convergence of η_i.

When $t\geq T_S$, one has $\tilde{S}_i=0$. Then, it can be obtained from (19) and (20) that

$$\dot{\tilde{\eta}}_i=S\tilde{\eta}_i-\bar{\varepsilon}_i, \tag{27}$$

and

$$\dot{\varepsilon}=(I_M\otimes S)\,\varepsilon-\left(\mathcal{L}_f\otimes \mathrm{I}_{n_0}\right)\bar{\varepsilon}. \tag{28}$$

Based on (7b), (21), and (23), the time derivative of V_2 along the trajectories of (28) satisfies

$$\begin{aligned}\dot{V}_2&=\bar{\varepsilon}^\top(\mathcal{D}\otimes S)\varepsilon-\frac{1}{2}\bar{\varepsilon}^\top\left(\mathcal{Q}\otimes \mathrm{I}_{n_0}\right)\bar{\varepsilon}+\frac{1}{2}\sum_{i=1}^M d_i(\varphi_i-\mu)\varepsilon_i^\top\varepsilon_i\\&\leq\frac{1}{2}(\|\mathcal{D}\otimes S\|^2\|\varepsilon\|^2-\|\bar{\varepsilon}\|^2)+\frac{1}{2}\sum_{i=1}^M d_i(\varphi_i-\mu)\varepsilon_i^\top\varepsilon_i.\end{aligned} \tag{29}$$

Note that $\bar{\varepsilon}_i=\varphi_i\varepsilon_i+c_2\mathrm{sign}^{\alpha_1}(\varepsilon_i)+c_3\mathrm{sign}^{\alpha_2}(\varepsilon_i)$. Since $\varphi_i\varepsilon_i$, $c_2\mathrm{sign}^{\alpha_1}(\varepsilon_i)$, and $c_3\mathrm{sign}^{\alpha_2}(\varepsilon_i)$ have component-wise sign consistency across all their respective elements, the terms in the expansion of $\|\bar{\varepsilon}_i\|^2$, including both square terms and cross terms, are non-negative. Then, one has

$$\|\bar{\varepsilon}\|^2\geq\sum_{i=1}^M\varphi_i^2\|\varepsilon_i\|^2+\sum_{i=1}^M c_2^2\|\varepsilon_i\|_{2\alpha_1}^{2\alpha_1}+\sum_{i=1}^M c_3^2\|\varepsilon_i\|_{2\alpha_2}^{2\alpha_2}. \tag{30}$$

Since $0<\alpha_1<1$ and $\alpha_2>1$, via Lemma 3, one has $\sum_{i=1}^M\|\varepsilon_i\|_{2\alpha_1}^{2\alpha_1}\geq\left(\|\varepsilon\|^2\right)^{\alpha_1}$ and $\sum_{i=1}^M\|\varepsilon_i\|_{2\alpha_2}^{2\alpha_2}\geq\frac{1}{(n_0M)^{\beta-1}}\left(\|\varepsilon\|^2\right)^{\alpha_2}$. It then follows that

$$\|\bar{\varepsilon}\|^2\geq\sum_{i=1}^M\varphi_i^2\|\varepsilon_i\|^2+c_2^2\left(\|\varepsilon\|^2\right)^{\alpha_1}+\frac{c_3^2\left(\|\varepsilon\|^2\right)^{\alpha_2}}{(n_0M)^{\beta-1}}. \tag{31}$$

Substituting (31) into (29) yields

$$\dot{V}_2\leq-\frac{1}{2}\sum_{i=1}^M\Gamma_i\varepsilon_i^\top\varepsilon_i-\frac{c_2^2\left(\|\varepsilon\|^2\right)^{\alpha_1}}{2}-\frac{c_3^2\left(\|\varepsilon\|^2\right)^{\alpha_2}}{2(n_0M)^{\alpha_2-1}}, \tag{32}$$

where $\Gamma_i\triangleq\varphi_i^2-d_i\varphi_i-\|\mathcal{D}\otimes S\|^2+d_i\mu$.

Select μ sufficiently large such that $\mu>\frac{d_{\max}}{4}+\frac{\|\mathcal{D}\otimes S\|^2}{d_{\min}}$. In this case, the discriminant of the quadratic equation Γ_i is negative, i.e., $\Delta_i\triangleq d_i^2-4(d_i\mu-\|\mathcal{D}\otimes S\|^2)<0$. Thus, we can obtain that $\Gamma_i\geq\Gamma_{i,\min}\triangleq d_i\mu-\|\mathcal{D}\otimes S\|^2-\frac{d_i^2}{4}>0$, $i=1,\dots,M$. Then, one has

$$\begin{aligned}\dot{V}_2&\leq-\frac{\Gamma_{\min}\|\varepsilon\|^2}{2}-\frac{c_2^2\left(\|\varepsilon\|^2\right)^{\alpha_1}}{2}-\frac{c_3^2\left(\|\varepsilon\|^2\right)^{\alpha_2}}{2(n_0M)^{\alpha_2-1}},\\&\leq-k_e\left(\|\varepsilon\|^2+\left(\|\varepsilon\|^2\right)^{\alpha_1}+\left(\|\varepsilon\|^2\right)^{\alpha_2}\right)\leq 0,\end{aligned} \tag{33}$$

where $\Gamma_{\min}\triangleq\min\{\Gamma_{1,\min},\dots,\Gamma_{M,\min}\}>0$ and $k_e=\min\{\Gamma_{\min},\frac{c_2^2}{2},\frac{c_3^2}{2(n_0M)^{\alpha_2-1}}\}>0$.

Since $0<\alpha_1<1$ and $\alpha_2>1$, via Lemma 3, one has $\sum_{i=1}^M\frac{c_2d_i}{1+\alpha_1}\|\varepsilon_i\|_{1+\alpha_1}^{1+\alpha_1}\leq\bar{c}_2(\|\varepsilon\|^2)^{\frac{1+\alpha_1}{2}}$ and $\sum_{i=1}^M\frac{c_3d_i}{1+\alpha_2}\|\varepsilon_i\|_{1+\alpha_2}^{1+\alpha_2}\leq\bar{c}_3(\|\varepsilon\|^2)^{\frac{1+\alpha_2}{2}}$, where $\bar{c}_2=(\frac{c_2d_{\max}}{1+\alpha_1})(n_0M)^{\frac{1-\alpha_1}{2}}$ and $\bar{c}_3=\frac{c_3d_{\max}}{1+\alpha_2}$. Note that $0<\frac{2\alpha_1}{1+\alpha_1}<1$ and $\frac{2\alpha_2}{1+\alpha_2}>1$. Then, by Lemma 3 again, one has

$$V_{21}^{\frac{2\alpha_1}{1+\alpha_1}} \leq k_{\alpha_1}\left((\|\varepsilon\|^2)^{\frac{2\alpha_1}{1+\alpha_1}} + (\|\varepsilon\|^2)^{\alpha_1} + (\|\varepsilon\|^2)^{\frac{\alpha_1(1+\alpha_2)}{1+\alpha_1}}\right), \tag{34}$$

$$V_{21}^{\frac{2\alpha_2}{1+\alpha_2}} \leq k_{\alpha_2}\left((\|\varepsilon\|^2)^{\frac{2\alpha_2}{1+\alpha_2}} + (\|\varepsilon\|^2)^{\alpha_2} + (\|\varepsilon\|^2)^{\frac{\alpha_2(1+\alpha_1)}{1+\alpha_2}}\right), \tag{35}$$

where $k_{\alpha_1} = \max\{(\Gamma_{\min})^{\frac{2\alpha_1}{1+\alpha_1}}, \bar{c}_2^{\frac{2\alpha_1}{1+\alpha_1}}, \bar{c}_3^{\frac{2\alpha_1}{1+\alpha_1}}\}$ and $k_{\alpha_2} = \max\{(\Gamma_{\min})^{\frac{2\alpha_2}{1+\alpha_2}}, \bar{c}_2^{\frac{2\alpha_2}{1+\alpha_2}}, \bar{c}_3^{\frac{2\alpha_2}{1+\alpha_2}}\} \times 3^{\frac{\alpha_2-1}{\alpha_2+1}}$. Since $\alpha_1 < \frac{2\alpha_1}{1+\alpha_1} < 1$ and $\alpha_1 < \frac{\alpha_1(1+\alpha_2)}{1+\alpha_1} < \alpha_2$, one has $(\|\varepsilon\|^2)^{\frac{2\alpha_1}{1+\alpha_1}} \leq \|\varepsilon\|^2 + (\|\varepsilon\|^2)^{\alpha_1}$ and $(\|\varepsilon\|^2)^{\frac{\alpha_1(1+\alpha_2)}{1+\alpha_1}} \leq (\|\varepsilon\|^2)^{\alpha_1} + (\|\varepsilon\|^2)^{\alpha_2}$. Similarly, since $1 < \frac{2\alpha_2}{1+\alpha_2} < \alpha_2$ and $\alpha_1 < \frac{\alpha_2(1+\alpha_1)}{1+\alpha_2} < \alpha_2$, one has $(\|\varepsilon\|^2)^{\frac{2\alpha_2}{1+\alpha_2}} \leq \|\varepsilon\|^2 + (\|\varepsilon\|^2)^{\alpha_2}$ and $(\|\varepsilon\|^2)^{\frac{\alpha_2(1+\alpha_1)}{1+\alpha_2}} \leq (\|\varepsilon\|^2)^{\alpha_1} + (\|\varepsilon\|^2)^{\alpha_2}$. It then follows that

$$V_{21}^{\frac{2\alpha_1}{1+\alpha_1}} \leq k_{\alpha_1}\left(\|\varepsilon\|^2 + 3(\|\varepsilon\|^2)^{\alpha_1} + (\|\varepsilon\|^2)^{\alpha_2}\right), \tag{36}$$

$$V_{21}^{\frac{2\alpha_2}{1+\alpha_2}} \leq k_{\alpha_2}\left(\|\varepsilon\|^2 + 3(\|\varepsilon\|^2)^{\alpha_2} + (\|\varepsilon\|^2)^{\alpha_1}\right). \tag{37}$$

Combining (36) and (37), one has

$$\frac{1}{4k_{\alpha_1}}V_{21}^{\frac{2\alpha_1}{1+\alpha_1}} + \frac{1}{4k_{\alpha_2}}V_{21}^{\frac{2\alpha_2}{1+\alpha_2}} \leq \left(\|\varepsilon\|^2 + (\|\varepsilon\|^2)^{\alpha_2} + (\|\varepsilon\|^2)^{\alpha_1}\right). \tag{38}$$

Substituting (38) into (33) yields

$$\dot{V}_2 \leq -\frac{k_e}{4k_{\alpha_1}}V_{21}^{\frac{2\alpha_1}{1+\alpha_1}} - \frac{k_e}{4k_{\alpha_2}}V_{21}^{\frac{2\alpha_2}{1+\alpha_2}}. \tag{39}$$

Note that by (7b), c_{1i} is positive due to $\dot{c}_{1i} \geq 0$ and $\hat{c}_{1i}(0) > 0$. Via (33), one can obtain that V_2, ε, and c_{1i} are bounded. Letting $\bar{c}_1 \triangleq \max_{i\in\mathcal{V}_f} c_{1i}$ and $k_c = \frac{d_{\max}(\mu+\bar{c}_1)}{2k_{\tilde{c}}}$, it then follows from (7b), (21), (25), and (39) that

$$\dot{V}_{21} = \dot{V}_2 - \dot{V}_{22} \leq -\frac{k_e}{4k_{\alpha_1}}V_{21}^{\frac{2\alpha_1}{1+\alpha_1}} - \frac{k_e}{4k_{\alpha_2}}V_{21}^{\frac{2\alpha_2}{1+\alpha_2}} + k_c V_{21}. \tag{40}$$

Let $\ell_1 = \left[\frac{(1-\theta_1)k_e}{4k_{\alpha_1}k_c}\right]^{\frac{1+\alpha_1}{1-\alpha_1}}$, where $\theta_1 \in (0,1)$. Define a bounded set $\Omega_1 = \{\varepsilon(t) \parallel V_{21}(\varepsilon(t)) \leq \ell_1\}$. The fixed-time convergence of η_i is analyzed under two conditions.

Condition 1: $\varepsilon(T_S) \in \Omega_1$

If $\varepsilon(T_S) \in \Omega_1$, it then follows from (40) that

$$\dot{V}_{21} \leq -\frac{k_e}{4k_{\alpha_1}}\theta_1 V_{21}^{\frac{2\alpha_1}{1+\alpha_1}} - \frac{k_e}{4k_{\alpha_2}}V_{21}^{\frac{2\alpha_2}{1+\alpha_2}}. \tag{41}$$

By Lemma 4, it follows from (41) that system (28) is fixed-time stable. The corresponding settling time $T_\eta \leq T_{\ell_1} \triangleq \frac{4k_{\alpha_1}(1+\alpha_1)}{k_e\theta_1(1-\alpha_1)} + \frac{4k_{\alpha_2}(1+\alpha_2)}{k_e(\alpha_2-1)} + T_S$.

Condition 2: $\varepsilon(T_S) \notin \Omega_1$

To proceed with the proof of the fixed-time convergence of η_i under this condition, the following claim is proved first.

Claim 1: There exists a bounded time $T^* > T_S$ such that $\varepsilon(T^*) \in \Omega_1$ and $\varepsilon(t) \in \Omega_1$ for all $t \geq T^*$.

This claim is proven by contradiction. Suppose the above conclusion is not true. Note that if $\varepsilon(T_S) \notin \Omega_1$, $V_{21}(\varepsilon(T_S)) > \ell_1$. Then, based on (39), the following inequality holds:

$$\begin{aligned} &V_2(\varepsilon(T_S)) \geq V_2(\varepsilon(T_S)) - V_2(\varepsilon(T^*)) \\ &\geq \int_{T_S}^{T^*}\left\{\frac{k_e}{4k_{\alpha_1}}V_{21}^{\frac{2\alpha_1}{1+\alpha_1}} + \frac{k_e}{4k_{\alpha_2}}V_{21}^{\frac{2\alpha_2}{1+\alpha_2}}\right\} ds \geq \phi(T^* - T_S), \end{aligned} \tag{42}$$

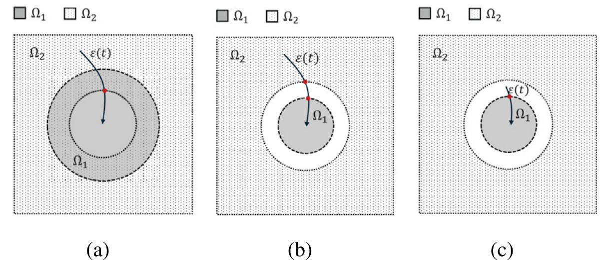


Fig. 1. Sketches of the sets Ω_1 and Ω_2. (a) Case 1. (b) Case 2. (c) Case 3.

where $\phi = \frac{k_e}{4k_{\alpha_1}}\ell_1^{\frac{2\alpha_1}{1+\alpha_1}} + \frac{k_e}{4k_{\alpha_2}}\ell_1^{\frac{2\alpha_2}{1+\alpha_2}}$.

From (42), one can conclude that $V_2(\varepsilon(T_S)) \to \infty$ as $T^* \to \infty$. This contradicts the fact that V_2 is bounded. Thus, there exists a bounded time $T^* > T_S$ satisfying $\varepsilon(T^*) \in \Omega_1$.

If $t \geq T^*$, via (41), one can obtain that the trajectory $\varepsilon(t)$ will stay in the set Ω_1 for all $t \geq T^*$. This completes the proof of Claim 1.

With this claim established, it follows from (41) that η_i is fixed-time convergent.

The rest gives the computation of the settling time T_η.

To proceed, define another set $\Omega_2 = \{\varepsilon(t) \parallel V_{21}(\varepsilon(t)) \geq \ell_2\}$, where $\ell_2 = \left[\frac{4k_{\alpha_2}k_c}{(1-\theta_2)k_e}\right]^{\frac{1+\alpha_2}{\alpha_2-1}}$ and $\theta_2 \in (0,1)$. If $\varepsilon(t) \in \Omega_2$, one has

$$\dot{V}_{21} \leq -\frac{k_e}{4k_{\alpha_1}}V_{21}^{\frac{2\alpha_1}{1+\alpha_1}} - \frac{k_e}{4k_{\alpha_2}}\theta_2 V_{21}^{\frac{2\alpha_2}{1+\alpha_2}}. \tag{43}$$

It follows from (43) that $V_{21}(t)$ is continuously decreasing. Then, the time interval that the trajectory $\varepsilon(t)$ reaches the boundary of Ω_2 can be upper bounded by $T_{\ell_2} \leq \frac{4k_{\alpha_1}(1+\alpha_1)}{k_e(1-\alpha_1)} + \frac{4k_{\alpha_2}(1+\alpha_2)}{k_e\theta_2(\alpha_2-1)}$.

Next, we will discuss the settling time in three cases respectively. Sketches of the sets Ω_1 and Ω_2 are shown in Fig. 1 for easy understanding. $\partial\Omega_1$ and $\partial\Omega_2$ denote the boundaries of Ω_1 and Ω_2, respectively.

Case 1: If $\ell_1 \geq \ell_2$, Fig. 1(a) illustrates that the trajectory of $\varepsilon(t)$ from $\varepsilon(T_S)$ to the origin involves two stages: (1) $\varepsilon(T_S) \to \partial\Omega_2$, and (2) $\partial\Omega_2 \to$ origin. Consequently, the settling time can be upper bounded by $T_\eta \leq T_{\ell_1} + T_{\ell_2}$.

Case 2: If $\ell_1 < \ell_2$ and $\varepsilon(T_S) \in \Omega_2$, Fig. 1(b) shows that the trajectory of $\varepsilon(t)$ from $\varepsilon(T_S)$ to the origin consists of three stages: (1) $\varepsilon(T_S) \to \partial\Omega_2$, (2) $\partial\Omega_2 \to \partial\Omega_1$, and (3) $\partial\Omega_1 \to$ origin. Not that $V_{21}(\varepsilon(t)) < \ell_2$ for $\varepsilon(t) \notin \Omega_2$ and $V_{22} \leq \bar{V}_{22} \triangleq \frac{M(\bar{c}_1+\mu)^2}{4}$. Then, via (42), the time interval of $\varepsilon(t)$ from $\partial\Omega_2$ to enter Ω_1 can be upper bounded by $T_\Delta \leq \frac{\ell_2+\bar{V}_{22}}{\phi}$. In this case, the settling time can be upper bounded by $T_\eta \leq T_{\ell_2} + T_\Delta + T_{\ell_1}$.

Case 3: If $\ell_1 < \ell_2$ and $\varepsilon(T_S) \notin \Omega_1 \cup \Omega_2$, Fig. 1(c) shows that the trajectory of $\varepsilon(t)$ from $\varepsilon(T_S)$ to the origin includes two stages: (1) $\varepsilon(T_S) \to \partial\Omega_1$, and (2) $\partial\Omega_1 \to$ origin. Then, the settling time can be upper bounded by $T_\eta \leq T_\Delta + T_{\ell_1}$, where T_Δ is given in Case 2.

In summary, based on the results obtained under two conditions, system (20) is fixed-time stable with T_η being the settling time. Given that $\varepsilon(t) = (\mathcal{L}_f \otimes I_{n_0})\eta(t)$ and $\mathcal{L}_f$ is nonsingular, it follows that $\lim_{t\to T_\eta}(\eta_i(t) - x_0(t)) = 0$ and $\eta_i(t) - x_0(t) = 0$ for $t \geq T_\eta$. This indicates the fixed-time convergence of $\eta(t)$.

Thus, the proof is completed. □

3.2. Adaptive fixed-time solution to the regulator equations

Our control strategy relies on solving the regulator Eqs. (5). Since only a subset of the followers has access to the knowledge of matrices S and F, the solution to this equation cannot be pre-computed for each follower. We propose to use $S_i(t)$ and $F_i(t)$ in

(7) to adaptively calculate the solution of the regulator equations in fixed time. In this case, inspired by Lemma 3 of Cai et al. (2017), the following lemma presents a method to calculate the solution to the regulator equations in fixed time.

Lemma 2. *Under Assumption 3, for any initial state $\hat{\varsigma}_i(0)$, the following equation,*

$$\begin{aligned}\dot{\hat{\varsigma}}_i(t) = &- \kappa\hat{\sigma}_i^{\top}(t)\text{sign}^{\varphi_1}(\hat{\sigma}_i(t)\hat{\varsigma}_i(t) - \hat{b}_i(t)) \\ &- \kappa\hat{\sigma}_i^{\top}(t)\text{sign}^{\varphi_2}(\hat{\sigma}_i(t)\hat{\varsigma}_i(t) - \hat{b}_i(t)),\end{aligned} \tag{44}$$

with $\hat{\sigma}_i(t) = S_i^{\top}(t) \otimes \begin{bmatrix} \text{I}_{n_i} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}\end{bmatrix} - \text{I}_{n_0} \otimes \begin{bmatrix} A_i & B_i \\ C_i & \mathbf{0}\end{bmatrix}$, $\hat{\varsigma}_i(t) = \text{vec}\left(\begin{bmatrix}\hat{X}_i(t) \\ \hat{U}_i(t)\end{bmatrix}\right)$, $\hat{b}_i(t) = \text{vec}\left(\begin{bmatrix}\mathbf{0} \\ -F_i(t)\end{bmatrix}\right)$, $\kappa > 0$, $0 < \varphi_1 < 1$, *and* $\varphi_2 > 1$, *has a unique bounded solution* $\hat{\varsigma}_i(t)$ *such that* $\hat{\varsigma}_i(t) \to \varsigma_i \triangleq \text{vec}\left(\begin{bmatrix}X_i \\ U_i\end{bmatrix}\right)$ *in a fixed time* T_ς. *Additionally, let* $\begin{bmatrix}\hat{X}_i(t) \\ \hat{U}_i(t)\end{bmatrix} = \text{M}^{n_0}_{n_i+m_i}(\hat{\varsigma}_i(t))$, *where* $i = 1, \ldots, M$, $\hat{X}_i(t) \in \mathbb{R}^{n_i\times n_0}$, $\hat{U}_i(t) \in \mathbb{R}^{m_i\times n_0}$. *It then holds that* $\hat{X}_i(t) \to X_i$, $\hat{U}_i(t) \to U_i$ *within the same fixed time* T_ς.

Proof. See Appendix B. □

3.3. Fixed-time FTC protocols

With the distributed fixed-time observer in (7) and the adaptive solution to the regulator equation in (44), we proceed to develop fixed-time FTC protocols in this part.

For each follower, the following fixed-time FTC protocol is proposed,

$$u_i(t) = \hat{U}_i(t)\eta_i(t) + U_{hi}h_i(t) + z_i(t),\ \ i \in \mathcal{V}_f, \tag{45}$$

where $\hat{U}_i$ and U_{hi} are the solutions of Eq. (44) and Eq. (5), respectively, $z_i(t)$ will be determined later.

Note that only those followers in direct communication with the leader have access to $x_0(t)$. For other followers, the fixed-time FTC component $z_i(t)$ should be designed based on the estimated state $\eta_i(t)$. Let $\hat{e}_i = x_i - \hat{X}_i\eta_i - X_{hi}h_i$. Given that (A_i, B_i) is controllable and $\text{rank}(B_i) = m_i$, it follows from Lemma 6 given in Appendix A that there exists a linear coordinate transformation $\hat{\zeta}_i = P_i\hat{e}_i = \left[\hat{\zeta}_{i1,1}, \ldots, \hat{\zeta}_{i1,\rho_1}, \ldots, \hat{\zeta}_{im_i,1}, \ldots, \hat{\zeta}_{im_i,\rho_{m_i}}\right]^{\top}$, where ρ_j, $j = 1, \ldots, m_i$, represents the controllability index of $\hat{\zeta}_{ij}$.

Then, $z_i(t)$ is proposed as follows:

$$z_i = M_i^{-1}(\omega_i - G_i\hat{e}_i), \tag{46}$$

where $\omega_i = [\omega_{i1}, \ldots, \omega_{ij}, \ldots, \omega_{im_i}]^{\top}$ with $\omega_{ij} = -\sum_{h=1}^{\rho_j}\left(k_{ij,h}\text{sign}^{p_{ij,h}}(\hat{\zeta}_{ij,h}) + \bar{k}_{ij,h}\text{sign}^{q_{ij,h}}(\hat{\zeta}_{ij,h})\right)$, G_i and invertible matrix M_i can be designed as in Lemma 6, the positive constants $k_{ij,h}$, $\bar{k}_{ij,h}$, $p_{ij,h}$, and $q_{ij,h}$, $j = 1, \ldots, m_i$, $h = 1, \ldots, \rho_j$, can be given as in Lemma 7 in Appendix A.

We are now prepared to present the main result of this subsection.

Theorem 2. *Consider the heterogeneous MAS (1) and (2) under Assumptions 1–4. The distributed fixed-time FTC problem under consideration is solved by the distributed control protocol consisting of (7), (44), (45), and (46).*

Proof. The proof comprises two parts. Firstly, we show that $\hat{e}_i$ is bounded in $[0, T_m)$, where $T_m \triangleq \max\{T_\eta, T_\varsigma\}$. Then, we establish that the formation tracking errors converge to zero in fixed time.

Part (i): Boundedness of $\hat{e}_i$ in $t \in [0, T_m)$.

Under Assumptions 3–4, from (1), (7), and (45), one has

$$\dot{\hat{e}}_i = A_i\hat{e}_i + B_iz_i + \Delta_{e_i}, \tag{47}$$

where $\Delta_{e_i} \triangleq B_i(\tilde{U}_i\eta_i + \tilde{U}_ix_0 + U_i\tilde{\eta}_i) + A_i(X_i\tilde{\eta}_i + \tilde{X}_ix_0 + \tilde{X}_i\tilde{\eta}_i) - \tilde{X}_i\eta_i - \tilde{X}_iSx_0 - \tilde{X}_i\tilde{\varepsilon}_i - X_i\tilde{\varepsilon}_i$, $\tilde{\varepsilon}_i \triangleq \bar{\varepsilon}_i + \tilde{S}_i\eta_i + S\tilde{\eta}_i$, $\tilde{U}_i = \hat{U}_i - U_i$, $\tilde{X}_i = \hat{X}_i - X_i$, and $\tilde{S}_i = \text{M}^{n_0}_{n_0}(\tilde{S}_{0i})$.

Let $\hat{\zeta}_i = P_i\hat{e}_i = \left[\hat{\zeta}_{i1,1}, \ldots, \hat{\zeta}_{i1,\rho_1}, \ldots, \hat{\zeta}_{im_i,1}, \ldots \hat{\zeta}_{im_i,\rho_{m_i}}\right]^{\top}$ and $\bar{\Delta}_{e_i} = P_i\Delta_{e_i} = \left[\bar{\Delta}_{i1,1}, \ldots, \bar{\Delta}_{i1,\rho_1}, \ldots, \bar{\Delta}_{im_i,1}, \ldots, \bar{\Delta}_{im_i,\rho_{m_i}}\right]^{\top}$. Then, the dynamics of $\hat{\zeta}_{ij}$, $j = 1, \ldots, m_i$, has the following form,

$$\begin{cases}\dot{\hat{\zeta}}_{ij,1} = \hat{\zeta}_{ij,2} + \bar{\Delta}_{ij,1}, \\ \dot{\hat{\zeta}}_{ij,2} = \hat{\zeta}_{ij,3} + \bar{\Delta}_{ij,2}, \\ \quad\vdots \\ \dot{\hat{\zeta}}_{ij,\rho_j} = \sum_{h=1}^{\rho_j}\left(k_{ij,h}\text{sign}^{p_{ij,h}}(\hat{\zeta}_{ij,h}) + \bar{k}_{ij,h}\text{sign}^{q_{ij,h}}(\hat{\zeta}_{ij,h})\right) + \bar{\Delta}_{ij,\rho_j}.\end{cases} \tag{48}$$

It is established in Tian et al. (2018) that system (48) is bounded-input-bounded-state (BIBS) stable, provided that $\bar{\Delta}_{ij} = [\bar{\Delta}_{ij,1}, \ldots, \bar{\Delta}_{ij,\rho_j}]^{\top}$ is considered as an input. Via Theorem 1 and Lemma 2, one has Δ_{e_i}, and consequently $\bar{\Delta}_{ij}$, $j = 1, \ldots, m_i$, are bounded in $[0, T_m)$. Then, one can derive that $\hat{\zeta}_i$ is bounded in $[0, T_m)$. Since $\hat{\zeta}_i = P_i\hat{e}_i$ and P_i is invertible, we can conclude that $\hat{e}_i$ is bounded in $[0, T_m)$.

Part (ii): Fixed-time convergence of $\hat{e}_i$.

When $t \geq T_m$, one has $\Delta_{e_i} = 0$. This implies that $\hat{e}_i = e_i \triangleq x_i - X_ix_0 - X_{hi}h_i$.

From Lemma 7, it can be concluded that $\forall j \in \{1, \ldots, m_i\}$, the system described by (48) is fixed-time stable, and the settling time is upper bounded by $T_o \leq T_m + T_\eta$, where T_η is determined based on Lemma 7. Since $\zeta_i(t) = P_ie_i(t)$, with P_i being invertible, and $e_{yi}(t) = C_ie_i(t) = y_i(t) - h_{oi}(t) - y_0(t)$, it follows that $\lim_{t\to T_o} e_{yi}(t) = 0$ and $e_{yi}(t) = 0$ for $t \geq T_o$, i.e., the fixed-time FTC problem under consideration is solved. Thus, the proof is completed. □

Remark 3. It is noted that the right-hand sides of Eqs. (7), (44), and (45) exhibit discontinuous at zeros. Consequently, the solutions to these equations are defined in the sense of Filippov (2013).

Remark 4. It can be observed from Theorem 2 that the formation tracking errors under the proposed control strategy converge to zero in fixed time with the upper bound of the settling time independent of the initial conditions of the system. However, it should be pointed out that the upper bound, though explicitly given, cannot be calculated due to its dependence on the adaptive gain which is adopted to avoid the global graph information and is unknown. Since the upper bound is explicitly given, even if it cannot be determined a priori, it is still possible to adjust certain design parameters to tune this upper bound. Moreover, it is important to note that the parameter $d_{\max}$ is only used in estimating the upper bound of the settling time, which means that the design and implementation of our proposed fixed-time control protocol do not rely on any global information.

Remark 5. In practical applications, not only the process of discretization but also other factors, such as system uncertainties and external disturbances, often make it infeasible for the system to achieve zero tracking error eventually. Thus, reducing tracking errors to a negligible level is usually sufficient in practice. In this case, our fixed-time method remains effective when applied in a discrete manner.

Remark 6. Compared with existing finite-time results (Cai et al., 2020; Fu & Wang, 2016; Wang et al., 2024, 2020, 2021; Xiao et al., 2009), the upper bound of the settling time in this paper is irrelevant to the initial conditions of the system. Moreover, by using the coordinate transformation method, the fixed-time controller is developed without the restrictive assumption on the full-row rank of the input matrix of the follower in Cai et al. (2022), Jiang et al. (2023), Zhang et al. (2021). Consequently, the potential application scope of this control protocol is significantly expanded. However, the proposed fixed-time control approach may require larger control inputs compared to those finite-time methods (Cai et al., 2020; Fu & Wang, 2016; Wang et al., 2024, 2020, 2021; Xiao et al., 2009), posing challenges for energy-limited tasks.

4. Simulation

Consider four practical cart-pendulums borrowed from Doyle et al. (2013) as followers of the following form,

$$\dot{x}_i = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{m_i g}{M_i} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{(M_i+m_i)g}{M_i l_i} & 0 \end{pmatrix} x_i + \begin{pmatrix} 0 & 0 \\ \frac{1}{M_i} & 0 \\ 0 & 0 \\ -\frac{1}{l_i M_i} & \frac{1}{l_i m_i} \end{pmatrix} u_i$$

$$y_i = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} x_i,$$

where $x_i = (p_i\ \dot{p}_i\ \psi_i\ \dot{\psi}_i)^\top$, $u_i = (f_i\ d_i)^\top$, $y_i = (p_i\ \psi_i)^\top$, p_i is the linear position of the cart i, ψ_i is the angle of pendulum i with respect to the vertical line, f_i is the horizontal force input to cart i, d_i is the perpendicular force input to pendulum i at the end, M_i and m_i are the masses of cart i and pendulum i, respectively, l_i is the length of pendulum i, and g represents the gravity constant with value of 9.8. The parameters $\{M_i, m_i, l_i\}$, $i = 1, 2, 3, 4$, are chosen as $\{1, 0.05, 60\}$, $\{1.1, 0.08, 61\}$, $\{1, 0.2, 61\}$, and $\{1.1, 0.3, 60\}$, respectively.

The system matrices of the leader are set as follows,

$$S = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The time-varying formation vector is chosen as $h_i(t) = \left(5\sin\left(t + \frac{(i-1)\pi}{2}\right)\quad 5\cos\left(t + \frac{(i-1)\pi}{2}\right)\right)^\top$, $i = 1, 2, 3, 4$, and $h_{oi}(t) = \left(5\sin\left(t + \frac{(i-1)\pi}{2}\right)\quad 0\right)^\top$. Solving the Eqs. (5) and (6) respectively yields

$$X_i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, U_i = \begin{pmatrix} -M_i & g m_i & 0 \\ -m_i & m_i l_i + \frac{g m_i l_i - g m_i (M_i + m_i)}{M_i} & 0 \end{pmatrix},$$

$$X_{hi} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}^\top, U_{hi} = \begin{pmatrix} -M_i & 0 \\ -m_i & 0 \end{pmatrix}.$$

The communication graph is given in Fig. 2. The parameters in the distributed fixed-time observer (7), adaptive regulator Eq. (44), and FTC protocol (46) are set as follows: $c_2 = c_4 = c_6 = \kappa = 5$, $c_3 = c_5 = c_7 = 6$, $\alpha_1 = \beta_1 = \gamma_1 = \varphi_1 = 7/9$, $\alpha_2 = \beta_2 = \gamma_2 = \varphi_2 = 907/700$, and $M_i = \mathrm{I}_2$, $i = 1, 2, 3, 4$. Let the initial conditions $x_i(0)$, $x_0(0)$, $\hat{S}_{0i}(0)$, $\hat{F}_{0i}(0)$, and $\hat{\varsigma}_i(0)$ are randomly generated within the range of -5 to 5. The other initial states are selected as $\eta_i(0) = \mathbf{0}_3$, $c_{1i}(0) = 1$.

The simulation results with one particular initial condition are presented in Figs. 3 and 4. The state observer errors between the distributed adaptive fixed-time observers and the leader are shown in Fig. 3(a). The adaptive gains of the observer are presented in Fig. 3(b), illustrating the convergence of all adaptive parameters to certain positive constants. The output formation tracking errors are shown in Fig. 4(a). It is noted that the formation tracking errors converge to the error tolerance threshold of 10^{-2} within 8.8s.

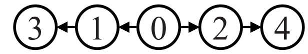
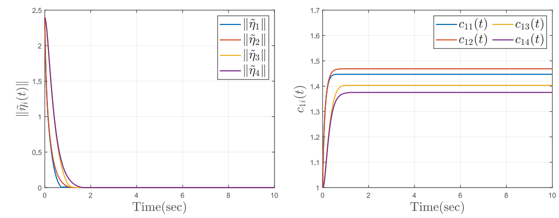
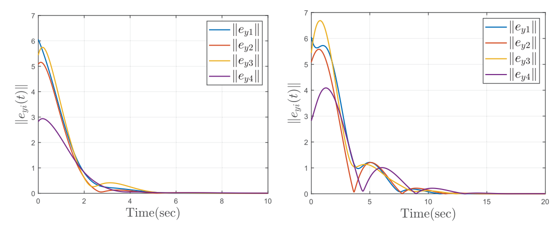


Fig. 2. Communication graph.



(a) (b)

Fig. 3. (a) The observer errors. (b) The adaptive observer gains.



(a) (b)

Fig. 4. (a) The formation tracking errors with the adaptive fixed-time controller. (b) The formation tracking errors with the adaptive finite-time controller.

For comparison, Fig. 4(b) shows the formation tracking errors using a finite-time controller derived by modifying our proposed control protocol to exclude the power index larger than one. Notably, the initial conditions remain the same. The comparison between Figs. 4(a) and 4(b) highlights that the fixed-time controller ensures a faster error convergence than its finite-time counterpart.

5. Conclusion

This paper addresses the problem of distributed adaptive fixed-time FTC for heterogeneous MASs. By developing a distributed adaptive fixed-time observer and adaptive fixed-time solution of the regulator equations, a fully distributed fixed-time FTC protocol has been proposed for each follower without requiring the restrictive full-row rank assumption. The proposed fixed-time control strategy successfully solves the FTC problem in a fully distributed manner, eliminating the need for global information. Simulations with comparisons have been conducted to validate the effectiveness of the proposed control protocol. Future research work can be focused on improving the proposed distributed adaptive fixed-time FTC method to eliminate the need for global information to estimate the upper bound of the settling time. Moreover, it is interesting to investigate the fixed-time sampled-data FTC problem for heterogeneous MASs.

Appendix A

Several key lemmas, which are utilized in the proof of the main results of this work, are introduced in this appendix.

Lemma 3 (Hardy et al., 1952). *For any $\xi_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, if $p \in (0, 1]$ and $q > 1$, the inequalities $\left(\sum_{i=1}^{n} |\xi_i|\right)^p \le \sum_{i=1}^{n} |\xi_i|^p \le n^{1-p}\left(\sum_{i=1}^{n} |\xi_i|\right)^p$ and $\sum_{i=1}^{n} |\xi_i|^q \le \left(\sum_{i=1}^{n} |\xi_i|\right)^q \le n^{q-1}\sum_{i=1}^{n} |\xi_i|^q$ hold.*

Lemma 4 (Polyakov, 2011). *Given the system $\dot{x} = f(x, t)$, where $f : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^n$ is a continuous vector function satisfying $f(0, t) = 0$. Suppose there exists a continuous, positive definite function $g(x) : \mathbb{R}^n \to \mathbb{R}$ such that $\dot{g}(x) \le -\kappa_c(g(x))^c - k_d(g(x))^d$ for all $x \in \mathbb{R}^n$, where $\kappa_c > 0$, $\kappa_d > 0$, $0 < c < 1$, and $d > 1$. The system is fixed-time stable, with an upper bound on the settling time $T \le \frac{1}{c_0(1-c)} + \frac{1}{d_0(d-1)}$.*

Lemma 5 (Qian & Lin, 2001). *For $x \in \mathbb{R}$ and $y \in \mathbb{R}$, if $a_1 > 0$ and $a_2 > 0$, $|x|^{a_1}|y|^{a_2} \le \frac{a_1|x|^{a_1+a_2}}{a_1+a_2} + \frac{a_2|x|^{a_1+a_2}}{a_1+a_2}$.*

Lemma 6 (Luenberger, 1967). *Consider a linear system*

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{49}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, A and B are the system matrices.

If (A, B) is controllable and the rank$B = m$, there exists a linear coordinate transformation $\zeta(t) = Qx(t) \in \mathbb{R}^n$ such that $\zeta(t) = \left[\zeta_1^\top(t), \ldots, \zeta_m^\top(t)\right]^\top$. Let ρ_i, $i = 1, \ldots, m$, denote the controllability index of $\zeta(t)$. Then, $\zeta(t)$ can be expressed as $\zeta(t) = \left[\zeta_{1,1}(t), \ldots, \zeta_{1,\rho_1}(t), \ldots, \zeta_{m,1}(t), \ldots, \zeta_{m,\rho_m}(t)\right]^\top$ and the dynamics of $\zeta_i(t)$, $i = 1, \ldots, m$, are given by:

$$\begin{cases} \dot{\zeta}_{i,1}(t) = \zeta_{i,2}(t), \\ \dot{\zeta}_{i,2}(t) = \zeta_{i,3}(t), \\ \vdots \\ \dot{\zeta}_{i,\rho_i}(t) = q_i^\top A^{\rho_i}x(t) + q_i^\top A^{\rho_i-1}Bu(t), \end{cases}$$

where $q_i^\top = i_{g_i}^\top Q$, $i_k \in \mathbb{R}^n$ is a vector with the kth element set to 1 and all the other elements set to 0, and $g_i = \sum_{k=0}^{i-1}(1 + \rho_k)$ with $\rho_0 = 0$.

Furthermore, defining $\zeta^\rho(t) = \left[\zeta_{1,\rho_1}(t), \ldots, \zeta_{m,\rho_m}(t)\right]^\top \in \mathbb{R}^m$, the dynamics can be described by,

$$\dot{\zeta}^\rho(t) = Gx(t) + Mu(t), \tag{50}$$

where $G = \begin{pmatrix} q_1^\top A^{\rho_1} \\ \vdots \\ q_m^\top A^{\rho_m} \end{pmatrix}$, and $M = \begin{pmatrix} q_1^\top A^{\rho_1-1}B \\ \vdots \\ q_m^\top A^{\rho_m-1}B \end{pmatrix}$ with M being invertible.

Lemma 7 (Basin et al., 2016). *Consider the system,*

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = x_3(t) \\ \vdots \\ \dot{x}_n(t) = u(t) \end{cases} \tag{51}$$

where $x(t) = [x_1(t), \ldots, x_n(t)]^\top$ and $u(t)$ represent the state and control input, respectively.

The fixed-time stabilization control input is given by:

$$u = -\sum_{i=1}^{n}\left(c_i \text{sign}^{\kappa_i}(x_i) + \bar{c}_i \text{sign}^{\bar{\kappa}_i}(x_i)\right), \tag{52}$$

where c_i and $\bar{c}_i$, $i = 1, \ldots, n$, are the positive constants of the polynomials $s^n + c_n s^{n-1} + \cdots + c_2 s + c_1$ and $s^n + \bar{c}_n s^{n-1} + \cdots + \bar{c}_2 s + \bar{c}_1$ which are required to be Hurwitz. The constants κ_i and $\bar{\kappa}_i$, $i = 1, 2, \ldots, n$, are chosen such that:

$$\kappa_{j-1} = \frac{\kappa_j\kappa_{j+1}}{2\kappa_{j+1} - \kappa_j}, \quad \bar{\kappa}_{j-1} = \frac{\bar{\kappa}_j\bar{\kappa}_{j+1}}{2\bar{\kappa}_{j+1} - \bar{\kappa}_j},$$

with $j = 2, \ldots, n$, $\kappa_{n+1} = \bar{\kappa}_{n+1} = 1$, $\kappa_n = \kappa \in (1 - \epsilon, 1)$, and $\bar{\kappa}_n = \bar{\kappa} \in (1, 1 + \bar{\epsilon})$, where $\epsilon, \bar{\epsilon} > 0$ are sufficiently small.

Then, the settling time is bounded by:

$$T \le \frac{p\lambda_{\max}(P_1)}{(1-p)\lambda_{\min}(Q_1)}\lambda_{\max}^{\frac{1-p}{p}}(P_1) + \frac{q\lambda_{\max}(P_2)}{(q-1)\lambda_{\min}(Q_2)}\lambda_{\max}^{\frac{q-1}{q}}(P_2), \tag{53}$$

where P_i, $i = 1, 2$, are real symmetric positive definite matrices that satisfy the Lyapunov equation $P_iA_i + A_i^\top P_i = -Q_i$, with Q_i being an arbitrary positive definite matrix. The matrices A_i, $i = 1, 2$, have the following forms:

$$A_1 = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -\kappa_1 & -\kappa_2 & \cdots & -\kappa_n \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -\bar{\kappa}_1 & -\bar{\kappa}_2 & \cdots & -\bar{\kappa}_n \end{pmatrix}.$$

Appendix B. Proof of Lemma 2

The proof of Lemma 2 is structured in two parts. Firstly, we demonstrate that $\hat{\varsigma}_i(t)$ remains bounded before the convergence of $\tilde{S}_{0i}(t)$ and $\tilde{F}_{0i}(t)$, i.e., $\hat{\varsigma}_i(t)$ remains bounded over the interval $[0, T_{SF})$, where $T_{SF} = \max\{T_S, T_F\}$. Then, we establish that $\hat{\varsigma}_i(t) \to \varsigma_i$ in fixed time.

Part (i): Show that $\hat{\varsigma}_i(t)$ is bounded in $[0, T_{SF})$.

Define the Lyapunov function candidate as $V_3 = \hat{\varsigma}_i^\top\hat{\varsigma}_i$. The time derivative of V_3 along the trajectory of (44) is obtained as

$$\begin{aligned} \dot{V}_3 &= -2\kappa\|\hat{y}_i\|_{1+\varphi_1}^{1+\varphi_1} - 2\kappa\|\hat{y}_i\|_{1+\varphi_2}^{1+\varphi_2} - 2\kappa\hat{b}_i^\top\text{sign}^{\varphi_1}(\hat{y}_i) - 2\kappa\hat{b}_i^\top\text{sign}^{\varphi_2}(\hat{y}_i) \\ &\le -2\kappa\|\hat{y}_i\|_{1+\varphi_1}^{1+\varphi_1} - 2\kappa\|\hat{y}_i\|_{1+\varphi_2}^{1+\varphi_2} + 2\kappa\sum_{j=1}^{(n_i+p)n_0}|\hat{b}_{ij}||\hat{y}_{ij}|^{\varphi_1} \\ &\quad + 2\kappa\sum_{j=1}^{(n_i+p)n_0}|\hat{b}_{ij}(t)||\hat{y}_{ij}(t)|^{\varphi_2}, \end{aligned} \tag{54}$$

where $\hat{y}_i = \hat{\sigma}_i\hat{\varsigma}_i - \hat{b}_i = \left[\hat{y}_{i1}, \ldots, \hat{y}_{i(n_i+p)n_0}\right]^\top$ and $\hat{b}_i = \left[\hat{b}_{i1}, \ldots, \hat{b}_{i(n_i+p)n_0}\right]^\top$.

Via Lemma 5 in Appendix A, one has $\sum_{j=1}^{(n_i+p)n_0}|\hat{b}_{ij}||\hat{y}_{ij}|^{\varphi_1} \le \sum_{j=1}^{(n_i+p)n_0}\frac{|\hat{b}_{ij}|^{1+\varphi_1}}{\varphi_1+1} + \frac{\varphi_1}{\varphi_1+1}\|\hat{y}_i\|_{1+\varphi_1}^{1+\varphi_1}$ and $\sum_{j=1}^{(n_i+p)n_0}|\hat{b}_{ij}||\hat{y}_{ij}|^{\varphi_2} \le \sum_{j=1}^{(n_i+p)n_0}\frac{|\hat{b}_{ij}|^{1+\varphi_2}}{\varphi_2+1} + \frac{\varphi_2}{\varphi_2+1}\|\hat{y}_i\|_{1+\varphi_2}^{1+\varphi_2}$. It then follows from (54) that

$$\dot{V}_3 \le -k_1\|\hat{y}_i\|_{1+\varphi_1}^{1+\varphi_1} - k_2\|\hat{y}_i\|_{1+\varphi_2}^{1+\varphi_2} + \hat{\varrho}_i, \tag{55}$$

where $k_1 = 2\kappa(1 - \frac{\varphi_1}{\varphi_1+1}) > 0$, $k_2 = 2\kappa(1 - \frac{\varphi_2}{\varphi_2+1}) > 0$, and $\hat{\varrho}_i = \sum_{j=1}^{(n_i+p)n_0}\left(\frac{|\hat{b}_{ij}|^{1+\varphi_1}}{\varphi_1+1} + \frac{|\hat{b}_{ij}|^{1+\varphi_2}}{\varphi_2+1}\right)$. Since $0 < \frac{1+\varphi_1}{2} < 1$ and $\frac{1+\varphi_2}{2} > 1$, via Lemma 3, one has $\|\hat{y}_i\|_{1+\varphi_1}^{1+\varphi_1} \ge (\|\hat{y}_i\|^2)^{\frac{1+\varphi_1}{2}}$ and $\|\hat{y}_i\|_{1+\varphi_2}^{1+\varphi_2} \ge ((n_i + p)n_0)^{\frac{1-\varphi_2}{2}}(\|\hat{y}_i\|^2)^{\frac{1+\varphi_2}{2}}$. Noting that $\frac{1+\varphi_1}{2} \le 1 \le \frac{1+\varphi_2}{2}$, one can derive that $\|\hat{y}_i\|^2 \le (\|\hat{y}_i\|^2)^{\frac{1+\varphi_1}{2}} + (\|\hat{y}_i\|^2)^{\frac{1+\varphi_2}{2}}$. Then, one has

$$\dot{V}_3 \le -k_y\left((\|\hat{y}_i\|^2)^{\frac{1+\varphi_1}{2}} + (\|\hat{y}_i\|^2)^{\frac{1+\varphi_2}{2}}\right) + \hat{\varrho}_i, \tag{56}$$

where $k_y = \min\{k_1, k_2((n_i + p)n_0)^{\frac{1-\varphi_2}{2}}\} > 0$.

Via Theorem 1, one has $\hat{\sigma}_i$ and $\hat{b}_i$ are bounded in $[0, T_{SF})$. We can find positive constants $\bar{\varrho}$, k_σ, and k_b satisfying $\hat{\varrho}_i \le \bar{\varrho}$ and $\|\hat{y}_i\|^2 = \|\hat{\sigma}_iQ_i\bar{\varsigma}_i - \hat{b}_i\|^2 \le k_\sigma\|\bar{\varsigma}_i\|^2 + k_b$ within this interval. Then, it follows that

$$\dot{V}_3 \le k_yk_\sigma V_3 + k_yk_b + \bar{\varrho}, \tag{57}$$

which implies that V_3, and consequently $\bar{\varsigma}_i(t)$ and $\hat{\varsigma}_i(t)$, are bounded in $[0, T_{SF})$.

Part (ii): Show that $\hat{\varsigma}_i(t) \to \varsigma_i$ in fixed time.

When $t \geq T_{SF}$, one has $\hat{\sigma}_i(t) = \sigma_i \triangleq S_i^\top \otimes \begin{bmatrix} I_{n_i} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} - I \otimes \begin{bmatrix} A_i & B_i \\ C_i & \mathbf{0} \end{bmatrix}$ and $\hat{b}_i(t) = b_i \triangleq \operatorname{vec}\left(\begin{bmatrix} \mathbf{0} \\ -F_i \end{bmatrix}\right)$. Following the result in Cai et al. (2017), there exists an orthogonal matrix $Q_i \in \mathbb{R}^{(n_0(n_i+m_i))\times(n_0(n_i+m_i))}$ such that $\sigma_i Q_i = [\bar{\sigma}_i\ \mathbf{0}]$, where $\bar{\sigma}_i \in \mathbb{R}^{n_0(n_i+m_i)\times r_i}$ with $\operatorname{rank}(\bar{\sigma}_i) = r_i$. Moreover, one can also find a unique constant vector $\varsigma_{i1} \in \mathbb{R}^{r_i}$ such that $\bar{\sigma}_i\varsigma_{i1} = b_i$.

Let $\bar{\varsigma}_i = Q_i^\top \hat{\varsigma}_i = [\bar{\varsigma}_{i1}^\top\ \bar{\varsigma}_{i2}^\top]^\top$, where $\bar{\varsigma}_{i1} \in \mathbb{R}^{r_i}$ and $\bar{\varsigma}_2 \in \mathbb{R}^{n_0(n_i+m_i)-r}$. When $t \geq T_{SF}$, the system (44) can be decomposed into:

$$\begin{aligned} \dot{\bar{\varsigma}}_{i1} &= -\kappa\bar{\sigma}_i^\top \operatorname{sign}^{\varphi_1}(\bar{\sigma}_i\bar{\varsigma}_{i1} - b_i) - \kappa\bar{\sigma}_i^\top \operatorname{sign}^{\varphi_2}(\bar{\sigma}_i\bar{\varsigma}_{i1} - b_i), \\ \dot{\bar{\varsigma}}_{i2} &= \mathbf{0}. \end{aligned} \tag{58}$$

It can be concluded from (58) that there is a constant vector $\varsigma_{i2} \in \mathbb{R}^{n_0(n_i+m_i)-r}$ such that $\bar{\varsigma}_{i2}(t) \to \varsigma_{i2}$ in the fixed time T_{SF}.

Let $\tilde{\varsigma}_{i1}(t) = \bar{\varsigma}_{i1}(t) - \varsigma_{i1}$. Noting that $\bar{\sigma}_i\varsigma_{i1} = b_i$, via (58), one has

$$\dot{\tilde{\varsigma}}_{i1} = -\kappa\bar{\sigma}_i^\top \operatorname{sign}^{\varphi_1}(\bar{\sigma}_i\tilde{\varsigma}_{i1}) - \kappa\bar{\sigma}_i^\top \operatorname{sign}^{\varphi_2}(\bar{\sigma}_i\tilde{\varsigma}_{i1}). \tag{59}$$

Consider the Lyapunov function candidate $V_4 = \tilde{\varsigma}_{i1}^\top\tilde{\varsigma}_{i1}$. Letting $\tilde{y}_i = \bar{\sigma}_i\tilde{\varsigma}_{i1}$, the time derivative of V_4 along the trajectory of (59) is given by

$$\dot{V}_4 = -2\kappa\|\tilde{y}_i\|_{1+\varphi_1}^{1+\varphi_1} - 2\kappa\|\tilde{y}_i\|_{1+\varphi_2}^{1+\varphi_2}. \tag{60}$$

Note that $0 < \varphi_1 < 1$ and $\varphi_2 > 1$. Via Lemma 3, one has $\|\tilde{y}_i\|_{1+\varphi_1}^{1+\varphi_1} \geq (\|\tilde{y}_i\|^2)^{\frac{1+\varphi_1}{2}}$ and $\|\tilde{y}_i\|_{1+\varphi_2}^{1+\varphi_2} \geq ((n_i+p)n_0)^{\frac{1-\varphi_2}{2}}(\|\tilde{y}_i\|^2)^{\frac{1+\varphi_2}{2}}$. Since $\bar{\sigma}_i^\top\bar{\sigma}_i$ is positive definite, it follows that $\|\tilde{y}_i\|^2 = \tilde{\varsigma}_{i1}^\top\bar{\sigma}_i^\top\bar{\sigma}_i\varsigma_{i1}^\top \geq \Lambda_{\min}V_4$, where $\Lambda_{\min} = \lambda_{\min}(\bar{\sigma}_i^\top\bar{\sigma}_i)$. Then, one has

$$\dot{V}_4 \leq -k_3V_4^{\frac{1+\varphi_1}{2}} - k_4V_4^{\frac{1+\varphi_2}{2}}, \tag{61}$$

where $k_3 = 2\kappa(\Lambda_{\min})^{\frac{1+\varphi_1}{2}}$ and $k_4 = 2\kappa(\Lambda_{\min})^{\frac{1+\varphi_2}{2}}((n_i+p)n_0)^{\frac{1-\varphi_2}{2}}$. Via Lemma 4, it can be obtained from (61) that $\bar{\varsigma}_{i1}(t) \to \varsigma_{i1}$ in a fixed time and the settling time can be estimated by $T_\varsigma \leq T_{SF} + \frac{2}{k_3(1-\varphi_1)} + \frac{2}{k_4(1-\varphi_2)}$.

Since $\bar{\varsigma}_i(t) = Q_i^\top\hat{\varsigma}_i(t) = [\bar{\varsigma}_{i1}^\top\ \bar{\varsigma}_{i2}^\top]^\top$ and Q_i is invertible, it then follows that $\hat{\varsigma}_i(t) \to \varsigma_i$ in a fixed time T_ς, where ς_i is a solution to $\sigma_i\varsigma_i = b_i$. According to Theorem 1.9 in Huang (2004), regulator Eqs. (5) can be reformulated as $\sigma\varsigma_i = b_i$. Thus, the adaptive regulator equations in (44) has a unique bounded solution $\hat{\varsigma}_i(t)$ such that $\hat{\varsigma}_i(t) \to \varsigma_i$, $\hat{X}_i(t) \to X_i$, and $\hat{U}_i(t) \to U_i$ in a fixed time T_ς. The proof is thus completed. □

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